Transport at Strong Coupling and Black Hole Dynamics

Vaios Ziogas

A Thesis presented for the degree of
Doctor of Philosophy

Centre for Particle Theory
Department of Mathematical Sciences
Durham University
United Kingdom

June 2018

Transport at Strong Coupling and Black Hole Dynamics

Vaios Ziogas

Submitted for the degree of Doctor of Philosophy
June 2018

Abstract:

In this thesis we study aspects of transport in strongly coupled quantum systems with broken translational symmetry. Using holographic duality, we also examine the associated dynamical problem in asymptotically Anti-de Sitter, spatially modulated black holes.

More precisely, in chapter 2 we consider the transport of conserved charges in spatially inhomogeneous quantum systems with a discrete lattice symmetry. When the DC conductivities are finite, we derive a set of generalised Einstein relations, relating the diffusion constants of the conserved charges to the DC conductivities and static susceptibilities. We also develop a long-wavelength expansion in order to explicitly construct the heat and charge diffusive modes within hydrodynamics on curved manifolds. In chapter 3 we used analogous techniques to construct the thermoelectric diffusive quasinormal modes in a large class of black hole spacetimes that are holographically dual to strongly coupled field theories in which spatial translations are broken explicitly. These modes satisfy a set of constraints on the black hole horizon, from which we find that their dispersion relations are given by the generalised Einstein relations. In chapter 4 we define a boost incoherent current in spontaneously modulated phases, and we show that in holographic theories, its DC conductivity can be obtained from solving a system of horizon Stokes equations.

Declaration

The work in this thesis is based on research carried out in the Department of Mathematical Sciences at Durham University. The results are based on the following collaborative works:

[1] A. Donos, J. P. Gauntlett, and V. Ziogas, "Diffusion in inhomogeneous media," *Phys. Rev.* **D96** no. 12, (2017) 125003, `arXiv:1708.05412 [hep-th]`.

[2] A. Donos, J. P. Gauntlett, and V. Ziogas, "Diffusion for Holographic Lattices," *JHEP* **1803** (2018) 056, `arXiv:1710.04221 [hep-th]`.

[3] A. Donos, J. P. Gauntlett, T. Griffin, and V. Ziogas, "Incoherent transport for phases that spontaneously break translations," *JHEP* **1804** (2018) 053, `arXiv:1801.09084 [hep-th]`.

During my Ph.D. I also published the following work, which is not included in this thesis:

[4] V. Ziogas, "Holographic mutual information in global Vaidya-BTZ spacetime," *JHEP* **1509** (2015) 114, `arXiv:1507.00306 [hep-th]`.

No part of this thesis has been submitted elsewhere for any degree or qualification.

Acknowledgements

This thesis is the culmination of my academic journey in the last five years. I would now like to thank some of the people who helped me along the way.

First and foremost, I would like to sincerely thank my Ph.D. supervisors. I was extremely fortunate to do most of my Ph.D. work with Aristomenis Donos. His devotion to physics, his wide range of knowledge, and his ability to spot interesting, well-posed problems, combined with his scientific integrity, make him a great supervisor and an inspirational role model. I also owe a lot of appreciation to Veronika Hubeny for her passion, patience and support, and for helping me stand on my own two feet. Sebastian Franco's enthusiasm was infectious and has stayed with me to this day.

Special thanks also go to Jerome Gauntlett; it is my joy and honour to be his collaborator. I am also extremely grateful to Christiana Pantelidou for many nice discussions on physics, and for always being available to help and advise.

I would also like to thank my officemates and all my fellow postgraduate students, including Matt Elliot-Ripley, Alex Peach, Richard Stewart, Xin Tang, Rebecca Bristow, and especially Sam Fearn and Hannah Harvey for their friendship all these years. I am grateful to Daniel Martin for useful discussions on numerics. Special thanks also go to my friend Benedict Aaronson; I will remember fondly our amusing discussions on physics, films and life in general.

I would also like to thank my fantastic family Nikos, Sophia and Ioanna. I would not be here without their unwavering love and encouragement. The constant supply of silly jokes made my life in Durham a little brighter every day.

Last but not least, I would like to thank my wonderful partner Maria, the funniest, kindest, loveliest person I have ever met! You made Durham feel like home.

My studies were supported by a Faculty of Science Durham Doctoral Scholarship.

We're all puppets, Laurie. I'm just a puppet who can see the strings.

— from *Watchmen* by Alan Moore

This thesis is dedicated
to

My grandparents

Βάγιας & Ελένη,
and
Ιωάννης & Βασιλική

Contents

Chapter 1

Introduction

Decades ago, Witten described string theory as “a piece of 21st-century physics that fell by chance into the 20th century” [5]. Indeed, even though it began as an attempt to provide a theory for the strong interactions by quantizing the flux lines between quarks, it was soon revealed that it naturally incorporates gravity in its spectrum [6][1]. Since then, there has been significant progress in understanding its physical and mathematical structure, making it the most succesful theory of quantum gravity to date.

Now, having had a clear view of early 21st-century physics, it would be fair to say that one of the dominant ideas has been the AdS/CFT correspondence, a conjectured equivalence between string/M-theory in a $(d+2)$-dimensional asymptotically Anti-de Sitter (AdS) spacetime and a conformal field theory (CFT) defined on its $(d+1)$-dimensional boundary[2]. Maldacena laid the foundations of the correspondence in 1997 [7] and the basic entries in the holographic dictionary were subsequently presented in [8, 9]. Heuristic ideas about holographic gravity theories, with degrees of freedom living on their boundary, had been around for a while [10, 11], but a concrete realisation required the mathematical structure of string theory and Maldacena’s deep physical insight.

Holography has since flourished into an extremely fruitful subject, residing at the intersection of high energy physics, condensed matter and quantum information theory. A particularly intriguing aspect is the weak/strong nature of the duality, allowing us to study strongly coupled field theories by solving classical gravitational problems, as well as gain insights into the emergence of spacetime geometry by using quantum information theory. This also means that holography is one of the most powerful tools one can use in order to model strongly interacting, non-quasiparticle

[1] In this introduction references will be mostly given to reviews and original or highly influential works.

[2] We are using the condensed matter convention in which d denotes the spatial dimensions of the field theory.

quantum matter in regimes where conventional techniques are not applicable. The goal of this thesis is to study some aspects of transport, in particular the diffusion of conserved charges, in strongly coupled systems, and the associated gravitational description, which involves the dynamics of black holes.

The structure of the introduction is as follows. In the first section 1.1 we introduce the holographic duality. We begin by discussing the origins of the AdS/CFT correspondence and the top-down and bottom-up approaches. We then present the basic entries in the holographic dictionary, as well as the radial Hamiltionian formalism for the gravity theory. We also explain how to incorporate temperature and chemical potential in holography, and we discuss the calculation of real time Green's functions. In the second section 1.2 we discuss the applications of gauge/gravity duality to condensed matter systems, usually referred to as "AdS/CMT". After some brief motivation, we discuss aspects of transport and hydrodynamics, and the relation to black hole quasinormal modes. We then move on to introduce symmetry breaking, focussing on breaking translations. We end with a discussion of thermoelectric conductivities and their calculations from holography.

1.1 Holography

1.1.1 AdS/CFT correspondence

Origins of the correspondence

In this subsection we will outline the original argument for AdS/CFT [7], which will provide important intuition for more general cases as well (see also the reviews [12–14]).

String theory originated as a quantum theory of strings propagating on some target spacetime. The need to include fermions in the spectrum and to eliminate a tachyonic mode led to the development of superstring theory[3]. Consistency of the quantum theory implies that quantum gauge and gravitational anomalies must cancel, and this produces five different $10-$dimensional superstring theories: type I, heterotic $SO(32)$, heterotic $E_8 \times E_8$, type IIA and type IIB. These (as well as the unique $11-$dimensional supergravity) are all connected via a complicated web of dualities, which is considered evidence for the existence of an underlying theory, called "M-theory", which includes all of the above theories in different perturbative regimes.

The massless spectrum of type IIA and type IIB superstring theories matches

[3]References to original works and more information on string theory can be found in the textbooks [6, 15–18].

exactly the spectrum of the corresponding supergravity theories. For instance, the bosonic sector of type IIB contains a graviton G Kalb-Ramond 2−form B_2 and a dilaton Φ (in the Neveu-Schwarz (NS) sector), as well as a scalar C_0, a 2−form C_2 and a 4−form C_4 with self-dual 5−form $F_5 = \star F_5$ (in the Ramond-Ramond (RR) sector). The strings can be closed or open; in the latter case their endpoints are restricted to $(p+1)-$dimensional hypersurfaces called "Dp branes". However, the Dp branes are more than just endpoints of strings: they are dynamical objects (described by a worldvolume DBI action), and they are charged under the RR fields. In the low energy limit, they are identified with the black p branes of supergravity. Thus, in modern superstring theory, the Dp branes (along with the NS5 branes and the M2, M5 branes and KK monopoles of M-theory) are on equal footing with the fundamental F1 strings.

Consider now a stack of N D3-branes in type IIB string theory. As mentioned above, two different pictures emerge for this configuration, depending on value of the parameter[4]

$$\lambda \equiv 4\pi g_s N, \tag{1.1.1}$$

where g_s is the string coupling constant.

- If $\lambda \ll 1$, the gravitational backreaction of the branes is negligible, and so we can treat them in the probe approximation as some light objects in 10d flat spacetime. Then, in the low energy limit $l_s \to 0$ (where l_s is the string length), the theory of open strings describing the brane dynamics reduces to 4d $\mathcal{N} = 4$ SYM with gauge group $SU(N)$. Note that this theory possesses an exact superconformal $SU(2,2|4)$ symmetry at the quantum level. λ is interpreted as the 't Hooft coupling, since we can identify

$$g_s = g_{YM}^2. \tag{1.1.2}$$

Additionally, the theory of closed strings in the bulk reduces to type IIB supergravity in the low energy limit, and these two sectors decouple.

- If $\lambda \gg 1$ the branes backreact on spacetime, giving rise to the following extremal black brane geometry and self-dual 5−form

$$\begin{aligned} ds^2 &= H^{-1/2}\left(-dt^2 + dx^i dx^i\right) + H^{1/2}\left(dr^2 + r^2 d\Omega_5\right), \\ F_5 &= dt \wedge dx^1 \wedge \cdots \wedge dx^4 \wedge H^{-1}, \qquad H = 1 + \left(\frac{L}{r}\right)^4, \end{aligned} \tag{1.1.3}$$

and the dilaton is constant. In the above r is the transverse radial coordinate, $d\Omega_5$

[4]Throughout this thesis we will be using units in which $c = \hbar = k_B = 1$.

is the standard metric on S^5 and

$$L^4 = \lambda \left(l_s\right)^4 \ . \tag{1.1.4}$$

The near horizon region $r \ll L$ is $\mathrm{AdS}_5 \times S^5$

$$ds^2 = \frac{L^2}{r^2} dr^2 + \frac{r^2}{L^2}\left(-dt^2 + dx^i dx^i\right) + L^2 d\Omega_5 \,, \tag{1.1.5}$$

with L being the radius of curvature of both AdS_5 and S^5. Now, from the point of view of an asymptotic observer, the near horizon string dynamics is effectively low energy due to the infinite redshift. At the same time, the local low energy excitations are described by type IIB supergravity. Again, these two sectors decouple in the low energy limit.

This decoupling argument led Maldacena to the remarkable conjecture that string theory in $\mathrm{AdS}_5 \times S^5$ with string coupling g_s and radius of curvature L is dual to $\mathcal{N} = 4$ SYM with gauge group $SU(N)$ and 't Hooft coupling λ. Following standard terminology, we will refer to the gravity theory as the "bulk theory" and to the QFT as the "boundary theory", since the latter can be thought of as living on the conformal boundary of the bulk AdS spacetime.

A more mathematical statement of the duality was presented in [8, 9] as an equality of partition functions:

$$Z_{string}\left[J\right] = Z_{SYM}\left[J\right]. \tag{1.1.6}$$

On the left hand we have the partition function of string theory in $\mathrm{AdS}_5 \times S^5$. On the right hand side we have the SYM partition function in the presence of sources J coupling to gauge invariant operators $\mathcal{O}$. Note that the quantization of string theory on curved spacetime is difficult, so the string partition function is not known; sometimes (1.1.6) is considered to be the definition of Z_{string}. In constrast, the SYM partition function in the presence of sources $Z_{SYM}\left[J\right]$ is a very common object in QFT: since SYM is a Lagrangian theory, it is given by the path integral

$$Z_{SYM}\left[J\right] = \int \mathcal{D}\Phi \exp\left[iS_{SYM} + i\int J\mathcal{O}\right], \tag{1.1.7}$$

where $S_{SYM} = \int \mathcal{L}_{SYM}\left[\Phi\right]$ is the SYM action, and by Φ we denote the fundamental fields of the theory. From this, we can compute correlation functions of gauge invariant operators by differentiating with respect to the sources and then setting them to zero.

This stringy, quantum version of the duality is usually referred to as the strong version of the AdS/CFT correspondence. There has been a lot of evidence that it is true [5]. However, we can also obtain various weaker limits of the duality. For example, taking $N \to \infty$ but keeping λ constant, we see from (1.1.1) that the string coupling g_s is small, and so the bulk theory reduces to tree-level string theory. The large N limit in the boundary gauge theory was studied by 't Hooft [19], who showed that it only involves planar Feynman graphs, with the $\mathcal{O}(1/N)$ corrections corresponding to higher genus graphs. On top of that, we can also vary the 't Hooft coupling λ. For small λ, the QFT is in its perturbative regime and from (1.1.4) we see that the bulk side is highly curved. However, the large λ limit has been by far the most fruitful: the bulk is weakly curved, and so the classical string theory reduces to classical supergravity, while the boundary theory is strongly coupled. This strong/weak nature of the AdS/CFT correspondence has been used to perform QFT calculations at strong coupling, inaccessible by other means, by doing classical gravity calculations in the dual theory. In this case, there is a $1-1$ correspondence between elementary bulk supergravity fields and boundary single-trace operators with small anomalous dimensions. This is the case which will concern us from now on.

In this limit, after analytically continuing to Euclidean signature and using the saddle point approximation, we can write (1.1.6) as follows:

$$-\,S^{sugra}_{on-shell}\left[\varphi \to J\right] = W_{CFT}\left[J\right]\,, \tag{1.1.8}$$

where W_{CFT} is the generating functional of connected correlation functions in the strongly coupled CFT and the set of bulk fields φ obeys the Dirichlet boundary conditions $\bar{\varphi}$ on the boundary of AdS (in an appropriate sense that we will describe below).

Top-down and bottom-up approaches

The two main approaches to the application of holographic duality are the "top-down" and the "bottom-up". In the former, one looks for a string/M-theory setup involving various branes and strings, and identifies the gauge theory living on their worldvolume. This is conjectured to be dual to quantum gravity in the near horizon geometry, much in the spirit of Maldacena's original argument. Apart from the prototypical 4d $\mathcal{N}=4$ super Yang-Mills $\Leftrightarrow$ string theory in $\mathrm{AdS}_5\times S^5$ discussed above, the most important examples of such dual pairs include 3d ABJM $\Leftrightarrow$ $\mathrm{AdS}_4\times S^7/\mathbb{Z}_k$ [20], 2d $\mathcal{N}=(4,4)$ SCFT $\Leftrightarrow$ string theory in $\mathrm{AdS}_3\times S^3\times M^4$ [7], an infinite class of

[5] As a first indication, one can easily check that the symmetries of both sides match: the isometry group of $\mathrm{AdS}_5\times S^5$ is $SO(4,2)\times SO(6)$, identical to the bosonic subgroup of the superconformal symmetry of the gauge theory. Similarly, the fermionic supersymmetries and superconformal symmetries also match [7].

4d $\mathcal{N}=1$ quiver gauge theories $\Leftrightarrow$ string theory in $\mathrm{AdS}_5\times SE^5$ [21] and 3d $\mathcal{N}=2$ Chern-Simons quiver gauge theories $\Leftrightarrow$ string theory in $\mathrm{AdS}_4\times SE^7$ [22] (by SE^n we denote toric Sasaki-Einstein $n-$manifolds).

Typically, constructions of this kind lead to geometries of the product form $\mathrm{AdS}_{\mathrm{d+2}}\times M^n$, with M^n a n-dimensional manifold, whose volume stays finite as we move towards the boundary of AdS, while the volume of the latter diverges. This "internal space" is very important in AdS/CFT; for example, it has the same global symmetries as the dual $\mathrm{CFT}_{\mathrm{d+1}}$ and its volume is related to the central charge [23]. One can always perform a dimensional reduction on M^n by decomposing all the fields in harmonics. Then, to each field in AdS corresponds an infinite tower of modes on M^n. All of this information is required to match the spectrum of the dual CFT[6]. However, in certain cases we can set a subset of these modes to zero, thus describing only a subsector of the CFT. In order to do this consistently, we have to show that the (generally coupled) equations of motion do not lead to constraints on the fields we keep turned on. The result of this "consistent truncation" is a theory in AdS whose solution can always be uplifted to a full solution of type IIB/11d supergravity.

The advantage of such top-down constructions is that we know many details of the dual field and so we can perform various controlled computations and non-trivial checks of the duality. However, if we would like to use holography in order to describe QCD or condensed matter systems at quantum critical points, we can use a more phenomenological, bottom-up approach. In that case, we write an effective bulk gravity theory incorporating the essential elements of the holographic dictionary that we want the field theory to possess. This leads to bulk theories with fewer fields and thus to greatly simplified calculations. However, we do not know the precise dual field theory or whether it can be consistently embedded in string theory. Despite all this, the hope is that the bottom-up model can still reveal universal properties, robust under possible modifications which will make the dual pair well defined. This approach, which is more generally called "gauge/gravity duality", is what we will have in mind in the rest of this thesis.

1.1.2 Elements of the holographic dictionary

In this subsection we are going to present the fundamental entries of the holographic dictionary, and expand in some detail on a few of them.

- The on-shell bulk action $S_{on-shell}$ is identified with the generating functional of connected correlation functions in the dual field theory W. Let us rewrite this

[6]In $\mathrm{AdS}_5/\mathrm{CFT}_4$ the matching has been done in detail, based on [24]. See also the nice explanation of the stringy exclusion principle [25] by the "giant gravitons" [26].

statement (1.1.8) in Lorentzian signature

$$iS_{on-shell}\left[\varphi \to J\right] = W_{CFT}\left[J\right]\ . \tag{1.1.9}$$

We can obtain the 1−point function of the operator $\mathcal{O}$ which couples to the source J by differentiating (1.1.9) with respect to J:

$$i\pi_\infty \equiv i\frac{\delta S_{on-shell}}{\delta J} = \frac{\delta W}{\delta J} \equiv \langle\, \mathcal{O}\,\rangle_J\, . \tag{1.1.10}$$

- There is a $1-1$ correspondence between bulk fields and boundary gauge invariant operators. The map is non-trivial, and is largely based on symmetry arguments. For example, the bulk metric $g_{\mu\nu}$ is dual to the stress-energy tensor of the dual field theory $T^{\mu\nu}$, a bulk $U(1)$ gauge field A_μ is dual to a conserved $U(1)$ current J^μ and a bulk scalar ϕ is dual to a scalar operator $\mathcal{O}$.

- Gauge symmetries in the bulk correspond to global symmetries on the boundary[7]. This can be seen by performing large gauge transformations and examining the asymptotic structure (this was known even before AdS/CFT, see [28, 29], and the review [30] and references therein).

- A classical background is dual to a quantum state of the dual QFT. The main examples are empty AdS (see (1.1.36) below) corresponding to the vacuum states, and AdS black holes corresponding to thermal states [31, 32] (see for example (1.1.37)).

- There is a UV/IR correspondence between the dual theories, in the sense that the long distance, IR region of the bulk theory describes the short distance, UV behavior of the field theory [33, 34]. Similarly, the deep interior of the bulk corresponds to the IR of the boundary theory. So, it is often said that holography geometrises the renormalization group flow of the field theory. This heuristic picture will provide useful intuition in the following, see also footnote 11 and [35].

- Equation (1.1.10) involves bare, divergent quantities. The LHS contains the usual UV divergencies of QFT and can be rendered finite by renormalization. The RHS contains IR divergencies, coming from the fact that the volume of asymptotically AdS spacetimes diverges as we move towards the boundary (see for example (1.1.11)). There is an analogous procedure in gravity which removes the divergent terms, called "holographic renormalization" [36].

[7]Note that a consistent theory of quantum gravity is not expected to have bulk global symmetries [27].

- The leading ("non-normalizable") asymptotic behavior of a bulk field determines the source of the dual operator and the subleading ("normalizable") behavior determines the expectation value ("VEV") of the dual operator [37] (except for the case of alternative quantization which we mention below). Let us sketch how this happens in the case of the metric, a scalar and a gauge field. We first choose Fefferman-Graham coordinates in which[8]

$$ds^2 = \frac{1}{\rho^2}\left(d\rho^2 + g_{\mu\nu}dx^\mu dx^\nu\right) . \tag{1.1.11}$$

Then, we can solve asymptotically the equations of motion coming from the bulk action (see (1.1.21)) obtaining the form [36]

$$g = g_{(0)} + \rho^2 g_{(2)} + \cdots + \rho^{d+1}\left(g_{(d+1)} + h_{(d+1)}\log\rho^2\right) + \mathcal{O}\left(\rho^{d+2}\right) . \tag{1.1.12}$$

Only even powers of ρ appear up to order $\lfloor d+1 \rfloor$. We also note that the metric diverges as $\rho \to 0$, and so it only defines a conformal structure at the boundary. The logarithmic term only appears for odd dimensions d. In the above series, the equations of motion do not determine $g_{(0)}$ and $g_{(d+1)}$, whereas $h_{(d+1)}$ and $g_{(n)}$ for $2 \leq n < d+1$ are functions of $g_{(0)}$ (the subleading terms are functions of both $g_{(0)}$ and $g_{(d+1)}$). Of course, it makes sense that the second order equations of motion require two boundary conditions; however, we will see later that we normally need to impose appropriate conditions at the interior of the bulk spacetime, and this will also determine $g_{(d+1)}$ in terms of $g_{(0)}$.

Now, we see that the holographic dictionary (1.1.9) interprets the Dirichlet boundary condition $g \to g_{(0)}$ as the field theory source coupling to the stress-energy tensor $T^{\mu\nu}$ [8, 9]. Using (1.1.10) and after performing a careful holographic renormalization analysis (which we discuss further near the end of this subsection), the 1−point function turns out to be [36, 38]

$$\langle\, T_{\mu\nu}\,\rangle = (d+1)g_{\mu\nu(d+1)} + X_{\mu\nu}[g_{(n)}]\, , \tag{1.1.13}$$

with $2 \leq n < d+1$ and $X_{\mu\nu}$ being a tensor whose form depends on the spacetime dimension.

Similarly, the asymptotic behavior of a scalar ϕ with mass m is:

$$\phi = \rho^{d+1-\Delta}\left(\phi_{(0)} + \rho^2\phi_{(2)} + \cdots + \rho^{2\Delta-d-1}\phi_{(2\Delta-d-1)} + \dots\right) , \tag{1.1.14}$$

[8]Here we are using standard notation and conventions with $\rho \to 0$ corresponding to the boundary and the subscripts in equations (1.1.12), (1.1.14), (1.1.17) denoting terms in the near boundary expansion, not to be confused with the notation in chapter 3 (see (3.2.5)).

where

$$m^2 = \Delta\left(\Delta - d - 1\right) \,. \tag{1.1.15}$$

Then, holographic renormalization leads to the 1−point function for the dual operator $\mathcal{O}$

$$\langle\, \mathcal{O}\, \rangle = (2\Delta - d - 1)\phi_{(2\Delta-d-1)} + \text{contact terms}\,. \tag{1.1.16}$$

For a $U(1)$ gauge field A_μ in the radial gauge $A_r = 0$ we find:

$$A_\mu = a_{\mu(0)} + \cdots + \rho^{d-1} a_{\mu(1)} + \dots\,, \tag{1.1.17}$$

leading to

$$\langle\, J^\mu\, \rangle = \frac{d-1}{e^2} a^\mu_{(1)}\,, \tag{1.1.18}$$

(see below (1.1.34) for the definition of e).

- By examining the scaling behavior of the dual operator $\mathcal{O}$, one can see that Δ as defined in (1.1.15) is the scaling dimension of the dual CFT operator. The unitarity bound in CFTs requires that $\Delta \geq (d-1)/2$, with the equality holding for free field theories (which are not relevant in holography). Note that for $(d-1)/2 < \Delta < (d+1)/2$ we can also choose the "alternative quantization", in which the source falls quicker than the VEV as we approach the boundary (1.1.14). In that case, performing a double trace deformation will induce a flow to a CFT with the standard quantization [39]. For $(d+1)/2 \leq \Delta < d+1$ the operator is relevant, in which case the bulk field does not spoil the AdS asymptotics (1.1.14). The marginal case is $\Delta = d+1$. From (1.1.12) and (1.1.17) one can see that the stress tensor operator is marginal and the global $U(1)$ current is relevant, as expected. Deformations by relevant or marginally relevant operators trigger RG flows to new IR fixed points. In contrast, irrelevant deformations with $\Delta > d+1$ destroy the asymptotics and the UV theory is not well defined; one needs to consider backreaction and find a new UV fixed point. However we will not consider this case; everywhere in the following the AdS asymptotics will correspond to CFT deformed by marginal or relevant deformations (i.e. the "asymptotically locally AdS" in the terminology of [40]).

We also note that from (1.1.15) we can derive the Breitenlohner-Freedman bound for the mass m:

$$m^2 \geq -\frac{(d+1)^2}{4}\,, \tag{1.1.19}$$

which was originally found by the requirement of perturbative stability [41, 42].

- Black hole quasinormal modes with normalizable conditions at the boundary and infalling conditions at the horizon correspond to poles of the retarded Green's function of the dual QFT. We will expand on this in subsection 1.2.2.

Radial Hamiltonian formulation

Let us now focus on the bulk side of the correspondence, which consists of a classical gravitational system. We will introduce the radial Hamiltonian formulation in order to discuss holographic renormalization in an elegant manner, and also to derive the constraints that we mention in subsection (1.2.4) and use extensively in chapter 3. For simplicity, we focus on the universal gravity sector, and we only briefly mention the generalisation to gravity-matter systems. Everything that we discuss here holds both in Euclidean and in Lorentzian signature; we will be using the latter in order to make contact with chapter 3. This discussion is based on [43, 44] and appendix A of [45]. For related work see [46–48].

We start by performing a radial Hamiltonian or "ADM" decomposition [49] by using a radial coordinate r as the Hamiltonian "time", with $r \to \infty$ corresponding to the (conformal) boundary of the spacetime[9]. We foliate our background in constant r hypersurfaces Σ_r, corresponding to the decomposition

$$ds^2 = \left(N^2 + N_\mu N^\mu\right) dr^2 + 2N_\mu dr dx^\mu + h_{\mu\nu}\, dx^\mu dx^\nu \,, \tag{1.1.20}$$

where $\mu, \nu, \dots = 1, \dots, d+1$. N and N_μ are the lapse and shift functions respectively, and $h_{\mu\nu}$ is the induced metric on Σ_r. The bulk action we are using is

$$S = \frac{1}{2\kappa^2} \int_M d^{d+2}x \sqrt{-g} \left(R + \frac{d(d+1)}{L^2} \right) + \int_{\partial M} d^{d+1}x \sqrt{-h}\, 2K \,. \tag{1.1.21}$$

where $\kappa^2 = 8\pi G_N$ with G_N being the $(d+2)$-dimensional Newton's constant. For convenience we will set $2\kappa^2 = 1$. L is the AdS radius, which we are free to set to 1, and we will do so from now on. The second term is the "Gibbons-Hawking term" [50], which we will discuss momentarily. Under the decomposition (1.1.20), the Ricci scalar R becomes

$$R = R[h] + K^2 - K_{\mu\nu}K^{\mu\nu} + \nabla_A \zeta^A \,. \tag{1.1.22}$$

Here $A, = 1, \dots, d+2$, ζ^A is a vector with $\zeta^r = -2K/N$, $R[h]$ is the Ricci curvature of the induced metric $K \equiv h^{\mu\nu}K_{\mu\nu}$, and $K_{\mu\nu}$ is the extrinsic curvature of Σ_r:

$$K_{\mu\nu} = \frac{1}{2N}\left(\dot{h}_{\mu\nu} - D_\mu N_\nu - D_\nu N_\mu\right) \,, \tag{1.1.23}$$

where the dot denotes derivatives with respect to r and D_μ is the covariant derivative

[9] Here we only use the fact that r is defined near the boundary. In chapter 3 we will be assuming that it extends until the black hole horizon, but the results hold more generally, see the discussion in [45]

with respect to $h_{\mu\nu}$.

Now, we note that the boundary term coming from the divergence of ζ cancels the Gibbons-Hawking term in (1.1.21). Ignoring the total spatial derivative, the action (1.1.21) becomes

$$S = \int dr \int_{\Sigma_r} d^{d+1}x \mathcal{L} \equiv \int dr \int_{\Sigma_r} d^{d+1}x \sqrt{-h}\, N \left[R[h] + K^2 - K_{\mu\nu}K^{\mu\nu} + d(d+1) \right] . \tag{1.1.24}$$

Let us denote the on-shell action as

$$S_{reg} = \int^r dr' \int_{\Sigma_r} d^{d+1}x \mathcal{L}|_{on-shell} . \tag{1.1.25}$$

This is a functional of the induced metric on Σ_r, since the bulk equations of motion are satisfied. The subscript is added to emphasise that this is a regulated form of the on-shell action; as already mentioned, S_{reg} will diverge in the near boundary limit $r \to \infty$.

We can now define the canonical momentum as usual

$$\pi^{\mu\nu} = \frac{\delta \mathcal{L}}{\delta \dot{h}_{\mu\nu}} = -\sqrt{-h} \left(K^{\mu\nu} - K h^{\mu\nu} \right) , \tag{1.1.26}$$

Note that this momentum is defined for any r. The notation in (1.1.9) indicates that π_∞ is obtained after taking the boundary limit $r \to \infty$ of π (and stripping of the appropriate powers of r, see (1.1.12), (1.1.14), (1.1.17)).

We can Legendre transform the Lagrangian density $\mathcal{L}$ to obtain the Hamiltonian density

$$\mathcal{H} \equiv \pi^{\mu\nu} \dot{h}_{\mu\nu} - \mathcal{L} = N\, H + N_\mu H^\mu , \tag{1.1.27}$$

where

$$H = -(-h)^{-1/2} \left(\pi_{\mu\nu}\pi^{\mu\nu} - \frac{1}{d}\pi^2 \right) - \sqrt{-h}\, \left(R[h] + d(d+1) \right) , \tag{1.1.28a}$$

$$H^\nu = -2\sqrt{-h}\, D_\mu \left((-h)^{-1/2} \pi^{\mu\nu} \right) , \tag{1.1.28b}$$

and $\pi \equiv \pi^\mu{}_\mu$. Now, N and N_μ act as Lagrange multipliers in (1.1.27) which impose the Hamiltonian and momentum contraints

$$H = 0 , \tag{1.1.29a}$$

$$H^\nu = 0 , \tag{1.1.29b}$$

and lead to the well-known fact that the bulk Hamiltonian vanishes in general relativity [44]. The constraints (1.1.29) are first class and they can be imposed on any constant-r hypersurface. Supplemented by the radial equations of motion for the conjugate momentum, they are equivalent to the second order Euler-Lagrange equations of motion.

In the case of the more general Einstein-Maxwell-Dilaton (see (1.1.34) below), one defines the canonical momenta for the scalar and the gauge field as in (1.1.26). This leads to a generalised form of the constraints (1.1.28), (1.1.29), as well as a new "current" constraint equivalent to the Gauss law [44]. Explicit expressions can be found in appendix 3.B.

Finally, let us note that the momentum constraint (1.1.29b) reflects diffeomorphism invariance on Σ_r. When evaluated at $r \to \infty$ and using (1.1.10), it leads to the conservation of the (bare) stress tensor of the dual field theory and similarly, the current constraint mentioned above leads to the conservation of the $U(1)$ current of the dual field theory. Indeed, the constraints generally lead to the (bare) field theory Ward identities [48, 51]. In view of chapter 3, we note that in the presence of arbitrary sources, they take the following form

$$\nabla^\mu \langle T_{\mu\nu} \rangle + \langle J^\mu \rangle F^{(0)}_{\mu\nu} + \langle \mathcal{O} \rangle \nabla_\nu \phi_{(0)} = 0\,, \tag{1.1.30a}$$

$$\nabla_\mu \langle J^\mu \rangle = 0\,, \tag{1.1.30b}$$

where ∇_μ is the covariant derivative with respect to $g_{(0)}$ and $F^{(0)}_{\mu\nu} = 2\partial_{[\mu} a^{(0)}_{\nu]}$ (see also (2.3.2)).

Holographic renormalization

The equation $\mathcal{H} = 0$, which is equivalent to (1.1.29), takes the Hamilton-Jacobi form once we express the canonical momentum in terms of Hamilton's principal function as in (1.1.10), since the latter can be identified with the on-shell action S_{reg}. Identifying the divergent local part of the solution of the Hamiltonian constraint (1.1.29a), we can add the corresponding local (on Σ_r) counterterms in order to renormalise the action,

$$S_{ren} \equiv S_{reg} + S_{ct}\,, \tag{1.1.31}$$

and we can safely take the limit $r \to \infty$ obtaining a finite result[10]. For the action (1.1.24) the counterterms were already identified by solving the equations of motion in a near boundary expansion in [36, 38]. For instance, for pure gravity it was found

$$\begin{aligned} S_{ct} &= \int_{\Sigma_r} d^2x\sqrt{-h}\left(2 - \frac{1}{2}R\log r\right)\,, && d = 1\,, \\ S_{ct} &= \int_{\Sigma_r} d^3x\sqrt{-h}\,(4 - 2R)\,, && d = 2\,, \end{aligned} \tag{1.1.32}$$

[10] One is also free to add local, finite scheme-dependent terms, but they will not concern us here.

with the logarithmic terms appearing in odd d related to the conformal anomaly [52] which comes from the choice of a particular representative in the conformal class of boundary metrics. For a free probe scalar we find

$$S_{ct} = \frac{d+1-\Delta}{2} \int_{\Sigma_r} d^{d+1}x \sqrt{-h}\, \phi^2 \,, \tag{1.1.33}$$

when $\Delta > (d+1)/2$. There are various systematic approaches for obtaining the counterterms for any bulk theory, for example one could solve the Hamilton-Jacobi equation recursively by expanding in eigenfunctions of the dilatation operator [43, 53]. See also [36, 54] for more general explicit expressions.

This process of removing the infinities on the gravity side is the aforementioned holographic renormalization. There is an analogy with the Gibbons-Hawking term, which is introduced in order to make the variational problem on compact manifolds with boundary well defined. Similarly, it can be shown that the holographic counterterms are necessary in order to make the variational problem well defined on asymptotically AdS manifolds [40] (and even in more general contexts [55, 56]).

There exists a direct correspondence between holographic renormalization in the Hamiltonian formalism with renormalization in QFT, if one expresses the latter as a Hamiltonian flow [43, 57][11]. A review of this exciting topic is outside the scope of this introduction. The main idea however is to substitute the renormalised bulk action S_{ren} in the divergent expressions (1.1.9), (1.1.10) and thus obtain finite, renormalised versions of the above expressions involving the renormalised momentum $\hat{\pi}$. After stripping off the appropriate powers of the radial coordinate r, one arrives at the expressions (1.1.13), (1.1.16), (1.1.18). We can similarly derive the Ward identities (1.1.30) for the renormalised VEVs.

1.1.3 Finite temperature and real time correlators

Finite temperature and chemical potential states

Let us now discuss some details about the relevant backgrounds. We will consider a class of bulk actions of the Einstein-Maxwell-Dilaton (EMD) form:

$$S = \int d^{d+2}x \sqrt{-g} \left(R - V(\phi) - \frac{Z(\phi)}{4} F^2 - \frac{1}{2} (\partial\phi)^2 \right) . \tag{1.1.34}$$

[11] The renormalization scale $\log\mu$ corresponds to the radial coordinate r (using coordinates in which the metric takes the form $ds^2 = dr^2 + h_{\mu\nu} dx^\mu dx^\nu$). Note however that there is not a precise relation between a radial cut-off in the bulk and a momentum cut-off in the boundary QFT, see [58, 59].

Here $V(\phi)$ and $Z(\phi)$ are functions of the scalar field ϕ, with $Z(0) = constant \equiv 1/e^2$ and

$$V(\phi) = -d(d+1) - \frac{m}{2}\phi^2 + \dots \,. \tag{1.1.35}$$

with m satisfying the BF bound (1.1.19).

Supplemented by a suitable set of Dirichlet boundary conditions, one needs to solve the equations of motion coming from (1.1.34) in order to obtain a background spacetime, which will be dual to a quantum state of the boundary QFT. The simplest case is empty AdS, with metric

$$ds^2 = -r^2\,dt^2 + \frac{1}{r^2}\,dr^2 + r^2\,dx^i dx^i\,, \tag{1.1.36}$$

and $A_\mu = \phi = 0$. Here r is a radial coordinate going from the Poincaré horizon $r = 0$ to the boundary $r \to \infty$. As we already mentioned, this background corresponds to the vacuum of the dual CFT on $\mathbb{R}^{d,1}$.

Another important example is the Reissner-Nordström-AdS solution

$$\begin{aligned} ds^2 &= -r^2 f(r)\,dt^2 + \frac{1}{r^2 f(r)}\,dr^2 + r^2\,dx^i dx^i\,, \\ A &= \mu\left[1 - \left(\frac{r_H}{r}\right)^{d-1}\right] dt\,, \qquad \phi = 0\,, \\ f(r) &= 1 - \left(1 - \frac{\mu^2}{r_H^2\gamma^2}\right)\left(\frac{r_H}{r}\right)^{d+1} + \frac{\mu^2}{r_H^2\gamma^2}\left(\frac{r_H}{r}\right)^{2d}\,, \quad \gamma^2 = \frac{2d}{d-1}e^2\,, \end{aligned} \tag{1.1.37}$$

which is a solution of (1.1.34) if $\partial_\phi Z(\phi)|_{\phi=0} = 0$. This is a black hole background with a bifurcate Killing horizon at r_H and an asymptotically AdS boundary at $r \to \infty$. From (1.1.17) and (1.1.18) we can see that this describes a CFT on flat space, deformed by a $U(1)$ chemical potential μ, with a charge density

$$\langle\, J^t\,\rangle = \frac{d-1}{e^2}\,r_H^{d-1}\,\mu\,. \tag{1.1.38}$$

By rotating to imaginary time, one can compute the relevant thermodynamic quantities [40, 60, 61], in the standard way from black hole thermodynamics (see [62–64] and the reviews [65, 66]). Firstly, the Euclideanised version of (1.1.37) is the usual cigar geometry ending at r_H. The absence of a conical singularity in the $it - r$ plane determines the temperature

$$T = \frac{r_H}{4\pi}\left(d + 1 - \frac{(d-1)\mu^2}{r_H^2\gamma^2}\right)\,, \tag{1.1.39}$$

and this means that the imaginary time is an S^1 with period $\beta \equiv 1/T$. The free energy is given by

$$F = -T \log Z = T S_E \,, \tag{1.1.40}$$

where S_E is the Euclidean renormalised bulk on-shell action, which depends on the whole bulk solution. From F we can obtain all the relevant thermodynamic quantities. For example, the entropy is computed from $S = (\beta\, \partial_\beta - 1)\, \beta F$, and we can show that it is given by the horizon area [67–69]. The energy is $E = -\partial_\beta\, (\beta F)$. By using Wald's formalism, one can rigorously show that the first law of thermodynamics $dE = T\, dS + \mu\, dQ$ holds [40].

Holography leads to the fascinating conclusion that the black hole thermodynamics described above can be identified with the thermodynamics of a field theory at finite temperature and chemical potential [31]. The topology of the boundary is $S^1 \times \mathbb{R}^d$ where the S^1 has periodicity β, as we expect from thermal field theory. Similarly, the Bekenstein-Hawking entropy is identified with the thermal entropy of the field theory, and the Hawking-Page transition in the bulk [70] is interpreted as a confinement-deconfinement phase transition (when the boundary is $S^1 \times S^3$)[31].

Returning to the Reissner-Nordström-AdS solution (1.1.37), we can see that it only depends on one tunable, dimensionless parameter, which we can take to be μ/T. In the $\mu \to 0$ limit we obtain the simpler Schwarzschild-AdS black hole solution, which describes a thermal state at temperature T (note that since the dual field theory is a CFT at finite temperature T, every non-zero T is equivalent; in the bulk this can be seen by rescaling the radial coordinate r). In the extremal $T \to 0$ limit we get a near horizon $\mathrm{AdS}_2{\times}\mathbb{R}^d$ geometry

$$ds^2 = L_2^2 \left[-r^2\, dt^2 + \frac{1}{r^2}\, dr^2 + \frac{(d-1)^2 \mu^2}{e^2}\, dx^i dx^i \right] , \qquad L_2^2 = \frac{1}{d(d+1)} \,, \tag{1.1.41}$$

where L_2 is the radius of curvature of AdS_2. This describes an emergent "local quantum criticality" with a scaling symmetry involving only the time coordinate. The discovery of this phase, which was shown to capture the low energy dissipative phenomena, was a milestone in the development of applied holography, [71, 72]. AdS_2 holography has gained much attention also due to the fact that it has non-zero entropy, violating the third law of thermodynamics. Initially this was considered a drawback, but the recently uncovered connections to the SYK model may suggest otherwise [73]. For $T \ll \mu$, we find a Schwarzschild-$\mathrm{AdS}_2{\times}\mathbb{R}^d$ spacetime, meaning that the horizon is entirely contained within the AdS_2 part of the spacetime.

Finally, we note that there is a more general class of finite temperature, finite chemical potential geometries than what we discussed above: one can consider a dynamic critical exponent $z > 1$ giving rise to the so-called "Lifshitz" geometries [74–77] (see also [78, 79]), as well as a hyperscaling violation exponent θ [80, 81], with a metric of the form

$$ds^2 = \left(\frac{r}{R}\right)^{d/(2\theta)} \left(-r^{2z}dt^2 + \frac{1}{r^2}\,dr^2 + r^2\,dx^i dx^i\right)\,, \tag{1.1.42}$$

where R is a constant of integration. These solutions are relevant from a condensed matter point of view, since for the general class of EMD actions (1.1.34), the above parametrisation in terms of z and θ is believed to be a complete classification of holographic strange metals at finite charge density (preserving translational invariance) [76, 80, 82].

Real time correlators

The main object of interest in the following will be the finite temperature 2-point retarded Green's functions, defined as

$$G_{AB}(t;t') = -i\theta(t-t')\,\langle\,[\mathcal{O}_A(t),\mathcal{O}_B(t')]\,\rangle_\beta\,, \tag{1.1.43}$$

for two local operators $\mathcal{O}_A$ and $\mathcal{O}_B$, where we have suppressed the spatial dependence and the expectation value is taken in a thermal state at temperature $T = 1/\beta$. Note that if we introduce a source for $\mathcal{O}_B$ in the Hamiltonian,

$$\delta H(t) = \delta J_B(t)\,\mathcal{O}_B(t)\,, \tag{1.1.44}$$

then the change in the expectation values of A is given by

$$\delta\langle\,\mathcal{O}_A\,\rangle(t) = \int dt'\,G_{AB}(t;t')\,\delta J_B(t')\,, \tag{1.1.45}$$

to linear order in δJ_B. This is the "linear response" regime, which describes perturbations around equilibrium states caused by the addition of small sources. The retarded Green's functions encode crucial information about the system: the on-shell, physical modes correspond to poles of the retarded Green's functions. Importantly, after Fourier transforming to momentum space, causality implies that $G_{AB}(\omega)$ is analytic for $\mathrm{Im}\omega > 0$; poles on the upper half complex plane lead to perturbative instabilities of the system.

We will give a self-contained presentation of the relevant details from linear response theory in section 2.2. In subsection 1.2.4 we will discuss the connection to transport which is of interest to condensed matter physicists; let us here describe the prescription for obtaining the retarded Green's functions from holography.

One of the main advantages of holography is that it is naturally formulated in Lorentzian signature (1.1.9). This means that we can directly compute real time correlation functions for strongly coupled field theories, where other techniques fail (for example, in lattice simulations the analytical continuation of Euclidean correlators from the discrete set of Matsubara frequencies is usually ambiguous, especially in the low frequency regime [83]; there is also the sign problem at finite chemical potential [84]).

Firstly, let us notice that Wick rotating to imaginary time is very useful in making sense of the saddle point approximation in the quantum gravity partition function (1.1.8), as well as in the abovementioned thermodynamic considerations. Apart from these, a Euclidean signature bulk leads to another simplification: as already mentioned, in solving the second order bulk equations of motion, we need to impose two "boundary" conditions. One of them is a Dirichlet condition for the source in (1.1.12), (1.1.14), (1.1.17). By analyzing the equations of motion in the deep interior (for example at the Poincaré or black hole horizon) in Euclidean signature we find that one of the two independent solutions diverges, while the other remains regular, making the choice of the regular one natural. However, in Lorentzian signature there are various conditions that can be imposed at the horizon, all of them regular, so it is not a priori clear which one to pick. We will return to this point momentarily.

Assuming that we choose an appropriate condition, this will determine the 1-point function of the dual operator as a function of the source, $\langle\, \mathcal{O}_A\,(J_B)\,\rangle$. As usual, in order to compute the 2-point correlation function we need to differentiate the above relation with respect to the source and then set $J_B = 0$. Thus, in order to compute the retarded 2-point function, we need to solve the linearised bulk equations of motion around a given background. G_{AB} will then be given by [85–87]:

$$G_{AB} = \left.\frac{\delta\langle\, \mathcal{O}_A\,\rangle}{\delta J_B}\right|_{\delta J_B=0} = \lim_{r\to\infty} r^{\alpha}\, \left.\frac{\delta\hat{\pi}_A}{\delta\varphi_B}\right|_{\delta J_B=0}\,, \tag{1.1.46}$$

where the power α is determined by requiring that the numerator gives the VEV and the denominator the source (we should only keep the finite piece in the above expression). For example, for scalar operators this becomes:

$$G_{AB} = (2\Delta_A - d - 1)\,\frac{\delta\phi_{A(2\Delta_A-d-1)}}{\delta\phi_{B(0)}}\,. \tag{1.1.47}$$

Let us now return to the issue of the choice of boundary conditions at the horizon. By examining the wave equation at the horizon one finds two possible behaviors, infalling and outgoing:

$$\phi_{in} \sim e^{-i\omega\, v_{EF}}\,, \qquad v_{EF} \equiv t + \ln\left(r - r_H\right)/4\pi T\,, \tag{1.1.48}$$

$$\phi_{out} \sim e^{-i\omega\, u_{EF}}\,, \qquad u_{EF} \equiv t - \ln\left(r - r_H\right)/4\pi T\,, \tag{1.1.49}$$

where v_{EF} and u_{EF} are the ingoing and outgoing Eddington-Finkelstein-like coordinates. The original prescription [88, 89] suggested that one can extract the retarded Green's functions from the on-shell action at the boundary after throwing away a horizon term, using the infalling conditions at the horizon. This was physically motivated from the fact that modes falling into the black hole describe dissipation from the dual field theory perspective[12]. Indeed, in addition to the black hole thermodynamics, the membrane paradigm [90, 91] also supports the fact that horizons are dissipative. Recall that the membrane paradigm suggests that we think of the black hole stretched horizon as a physical fluid membrane with usual transport properties, obeying the Navier-Stokes equations. These ideas provide profound connections between thermal field theories and black hole phenomena.

The above prescription has been justified in a number of ways. [92] derived it by using the maximally extended Schwarzschild-AdS spacetime, which [32] showed that it is dual to a tensor product of two CFTs. They identified the second CFT as the copy of the theory used in the Schwinger-Keldysh (S-K) formalism [93, 94]. This way they were able to compute the various S-K correlators, and argue that the natural infalling conditions come from the S-K contour time ordering. Additional supporting evidence was given in the same paper, by showing that at zero temperature, infalling conditions follow from analytic continuation of the Euclidean Green's function. [86] gave a similar argument for the finite temperature case.

Finally, [95, 96] gave a more general prescription for the computation of real time correlators. The basic idea is that the RHS of (1.1.9) involves integrating over a contour on the complex time plane in the S-K formalism. Thus, one should "fill it in" with the bulk solution of the LHS. This leads to a bulk metric with mixed signature. Using appropriate continuity conditions in the matching hypersurfaces, one is in principle able to directly compute any real time correlator. It was shown in [96] and more explicitly in [97] that this also leads to infalling conditions for the retarded Green's function.

Before closing this section, let us mention that higher point correlation functions have also gathered much attention lately. More specifically, [98–100] connected out-of-time-order 4-point correlation functions to quantum chaos in the dual field theory, and suggested a holographic computation using shock-wave geometries. Comparisons with field theory results in simple cases [101], interesting bounds [102] and relations to hydrodynamics [103–108] indicate that there are still many discoveries to be made in this exciting area.

[12] Similarly, the outgoing modes give the advanced Green's function.

1.2 Applications

1.2.1 AdS/CMT

The conventional theories for understanding most of the condensed matter phenomena, including metallic transport, thermal phase transitions and superconductivity, were developed in the last couple of centuries and have been extremely successful. Most of them rely on the existence of some effective, weakly coupled quasiparticle description of the system under investigation. For example, this can be the electrons and phonons in metals, the Cooper pairs in superconductors or the order parameters in phase transitions (a review of such methods is outside the scope of this introduction; we refer the interested reader to [109–111] for more details).

However, over the past decades, there has been increasing evidence supporting the existence of strongly coupled systems without quasiparticles [112]. For instance, such "strange metallic" phases have been found in various compounds such as cuprates and pnictides. The fact that these metals do not admit any quasiparticle description can be inferred from the linear in temperature resistivity [113], violating the Mott-Ioffe-Regel (MIR) bound [114, 115] and the Wiedemann-Franz law [116, 117], and exhibiting non-Drude-like low frequency behavior in their DC conductivity (see equation (1.2.8)) [109]. Thus, such strongly coupled strange metals elude description by the conventional Fermi liquid theory.

Another curious phenomenon is high-temperature superconductivity. The normal phase of a conventional superconductor consists of a (weakly coupled) metallic Fermi liquid for $T > T_c$, where T_c is the critical temperature. The superconducting phase appears at $T < T_c$ and is characterised by condensation of electrons into Cooper pairs. This mechanism is the basis of the Bardeen-Cooper-Schrieffer (BCS) theory of superconductivity [118] and it predicts critical temperatures up to approximately $30K$. However, superconducting phases with significantly higher critical temperature have been experimentally observed [119]. Strange metals appear to constitute the normal phase of high-T_c superconductors, and so understanding their properties is imperative in formulating a theory of high-T_c superconductivity.

Additionally, we would like to understand quantum phase transitions (QPT) such as metal-insulator transitions[13], which are phase transitions driven by quantum fluctuations. The standard way of characterizing continuous phase transitions is the universal Landau-Ginzburg-Wilson theory [109]. Using an effective field theory approach, we are instructed to identify the symmetries of the various phases. Then, phase transitions are described by the spontaneous breaking of a subset of these

[13] We remind the reader that insulators are systems with vanishing DC conductivity $\sigma_{DC} \to 0$ as $T \to 0$.

symmetries, and the relevant order parameter specifies the dynamics. At zero temperature, QPTs take place at a value g_c of a tunable parameter g. Upon taking $g \to g_c$, the mass gap normally behaves as $\Delta \sim (g - g_c)^{\nu z}$, while the correlation length behaves as $\xi \sim (g - g_c)^{-\nu}$. The critical point is characterised by the dynamical critical exponent z specifying the scaling of time relative to space, which we also encountered in subsection 1.1.3. When $z = 1$, the scaling symmetry combines with the Lorentz symmetry, making the description of the critical point by a CFT possible. In addition, we expect that the quantum critical point may extend its influence to a "quantum critical region" of the phase diagram with non-zero temperatures $T > 0$ [109, 119]. Such phases involve long range quantum entanglement and can be generally modelled by emergent, strongly coupled, CFTs.

The QFT techniques for studying strongly coupled theories are limited, although some general statements can be made with memory matrix-like arguments (see 2.2) as well as within the framework of hydrodynamics (see the following subsection (1.2.2) and section (2.3)). This is where holography comes into play, since it can transform a hard QFT problem into a much easier classical gravitational problem. See [72, 87, 120–127] for various books and reviews dedicated to the application of holography to condensed matter systems.

We stress from the outset that holography is meant to be an effective description of strongly coupled condensed matter systems. This addresses the concern that in gauge/gravity duality the UV theory is usually highly symmetric (conformally invariant, supersymmetric), as happens in AdS/CFT. However, we have already seen ways to break such unrealistic symmetries: introducing finite temperature, or sources for relevant operators such as chemical potentials or holographic lattices (see subsection 1.2.3), the UV theory gets deformed, resulting in a flow to a less symmetric IR ground state. Finally, one can also consider non-relativistic asymptotics (see equation (1.1.42)). A more significant point is the fact that we require the QFT to be in the large N limit in order for the bulk theory to be classical, which generally suppresses quantum fluctuations. Despite the above issues, the hope is that holography may be albe to capture universal IR properties of strongly coupled systems [128, 129].

In view of condensed matter applications, we will focus on boundary theories with a global $U(1)$ symmetry. In conventional metals, this comes from the screening of Coulomb interactions. This turns out to be useful even at strong coupling, where we think as having already integrated out the dynamics of the electromagnetic $U(1)$ gauge symmetry[14]. From this effective point of view, it is now apparent why we

[14]In top-down constructions, it can be obtained as a subgroup of the R-symmetry of the boundary theory.

focussed on the class of EMD bulk actions (1.1.34): it allows us to obtain universal results for strongly coupled QFTs with global $U(1)$ symmetry, while the dilaton can model a general scalar order parameter.

1.2.2 Hydrodynamics and black hole quasinormal modes

Transport coefficients

As already mentioned in subsection (1.1.3), holography is well suited for studying out of equilibrium phenomena in real time. The most interesting quantities, both from an experimental and a theoretical point of view, are the transport coefficients, such as the conductivities, diffusion constants and viscosities. These are defined within the regime of linear response, they depend on the microscopic QFT and they characterise the transport of conserved charges [130, 131].

For simplicity, in this subsection we assume that our system preserves translational symmetry. Then we can Fourier transform to momentum space parametrised by the frequency ω and the spatial momentum $\mathbf{k}$. Using the fundamental relation (1.1.45) we can derive the microscopic definition of some transport quantities and static susceptibilities. For instance, for systems with a global $U(1)$ symmetry, the thermoelectric conductivities are defined by a generalisation of Ohm's law:

$$\begin{pmatrix} J^i \\ Q^i \end{pmatrix} = \begin{pmatrix} \sigma^{ij} & T\alpha^{ij} \\ T\bar{\alpha}^{ij} & T\bar{\kappa}^{ij} \end{pmatrix} \begin{pmatrix} E_j \\ \zeta_j \end{pmatrix}, \tag{1.2.1}$$

where $Q^i = -T^i_t - \mu J^i$ is the heat current, which couples to a homogeneous thermal gradient parametrised by ζ_j, and J^i is the $U(1)$ current, which couples to the homogeneous electric field E_j. By comparing with (1.1.45), we can derive Kubo's formula[15]

$$\sigma^{ij}(\omega) = -\lim_{\mathbf{k}\to 0} \frac{1}{i\omega} G_{J^iJ^j}(\omega, \mathbf{k}), \tag{1.2.2}$$

and similarly for the other components of the conductivity matrix (1.2.1). In the same way we arrive at Kubo's formula for the shear viscosity

$$\eta = \lim_{\omega\to 0} \frac{1}{\omega} \mathrm{Im} G_{T^{ij}T^{ij}}(\omega, \mathbf{0}). \tag{1.2.3}$$

These relations capture the essence of the fluctuation-dissipation theorem [131]. A more complete discussion (focussed on the conductivities and diffusion constants) can be found in section 2.2.

In the zero frequency limit, the above "AC" or "optical" conductivities are called

[15] We are using sign conventions consistent with chapter 2, see appendix 2.A.

DC conductivities; for example the electric DC conductivity is defined by

$$\sigma^{ij}_{DC} \equiv \lim_{\omega\to 0} \sigma^{ij}(\omega)\,. \tag{1.2.4}$$

These transport quantities are of particular interest, since they describe the response of the system to a constant thermal or electric source. Importantly, when translations are preserved and the background charge density is non-zero $\rho \neq 0$, the DC conductivities $\sigma_{DC}, \alpha_{DC}, \bar{\alpha}_{DC}$ and $\bar{\kappa}_{DC}$ will all diverge (as in equation (4.2.12)) [132]. This can be argued using the Ward identities, which relate the various thermoelectric conductivities as in equation (4.2.7), see [124, 133]. Another way to see the divergence is by performing a boost, since this will transform the currents leading to non-uniqueness of the conductivities[16]. The latter approach will play an important role in generalising the above results to the case of spontaneously broken translational invariance, see chapter 4. More generally, when a conserved operator overlaps with the current operator, the corresponding DC conductivities diverge [123]. Effectively, the conserved or nearly conserved operator forces all operators which overlap with it to decay "coherently", i.e. at its own time scale.

In contrast, in charge neutral systems the electric current does not overlap with momentum, and thus we get a finite electric DC conductivity σ_{DC} (while the thermal conductivity $\bar{\kappa}_{DC}$ still diverges). It turns out that even in systems at finite charge density, we can define an "incoherent" current which does not overlap with the momentum and which has a finite conductivity [134]; this is the subject of chapter 4.

Finally, note that discrete symmetries impose constraints on the conductivities. For instance, systems with time reversal and reflection symmetry satisfy the Onsager relations $\sigma = \sigma^T$, $\bar{\kappa} = \bar{\kappa}^T$, $\alpha^T = \bar{\alpha}$ [124, 131, 135].

We will discuss the conductivities further in subsection 1.2.4.

Hydrodynamics

Let us now sketch the basic ideas of hydrodynamics. This is a huge subject with a wide variety of applications, but here we only intend to introduce its philosophy; some explicit expressions can be found in section 2.3, see also [136–140] for complete discussions.

Hydrodynamics is a framework to describe QFTs as they approach equilibrium. It involves a local thermal equilibration scale τ_{eq}, after which we assume that our

[16] An equivalent but more heuristic argument is the following: assuming that the momentum carriers also carry $U(1)$ charge, by boosting the system we can generate a non-trivial electric current without the application of external electric source; then the definition (1.2.1) implies that the DC conductivity diverges.

system is described by a reduced set of dynamical degrees of freedom: instead of the full stress-tensor $T^{\mu\nu}$ and $U(1)$ current operator J^μ, we consider a set of local fields including the temperature $T(x)$, the normalised four-velocity $u^\mu(x)$ and the chemical potential $\mu(x)$[17]. Generically, all other operators will relax within τ_{eq}, but the conservation laws of $T^{\mu\nu}$ and J^μ (i.e. the Ward identities (1.1.30)) imply that they will be relevant at long wavelengths and late times. So, this effective description is expected to hold as long as the variation of the fields is slow compared to the local thermal equilibration length $l_{eq} \sim T$.

We can then expand $T^{\mu\nu}$ and J^μ in gradients of the thermodynamic variables, resulting in the so-called "constitutive relations", see equation (2.3.3). At zero order we get an ideal fluid, involving thermodynamic quantities such as the energy ϵ, the pressure P and the charge density ρ. At first order we get dissipation, and the coefficients in the series expansion are various transport coefficients, such as the "quantum critical" or "incoherent" conductivity σ_Q (we will discuss this quantity in more detail in subsection 1.2.4 and chapter 4), and the shear and bulk viscosities η and ζ_b, as in equation (2.3.3). These constitute the only information which is determined by the underlying QFT within the framework of hydrodynamics.

The hydrodynamic modes of the system have a dispersion relation $\omega(\mathbf{k})$ such that $\lim_{\mathbf{k}\to 0} \omega(\mathbf{k}) = 0$, or, in other words, they are "gapless". We can obtain these modes by going to the linear response regime of hydrodynamics, i.e. linearising the conservation equations around equilibrium (in which case we will also refer to the latter as the "Navier-Stokes" equations with some abuse of terminology). For example, in the case we are considering here where translations are preserved, there is a transverse diffusive mode, a longitudinal diffusive mode [134], and a sound mode. Additionally, one can compute the current-current retarded Green's functions from hydrodynamics using the Kadanoff-Martin method, see [137, 142] and appendix 2.B.

A milestone in the development of applied holography was achieved with the realisation that this hydrodynamic derivative expansion can also be performed in the dual gravity theory [143–145] (see also the reviews [138, 146, 147]). Roughly, one can take the boosted black brane bulk solution and consider making some fields space dependent. Then, one can demand that this is a solution of the bulk equations of motion order by order in a long-wavelength expansion, and this reproduces the constitutive relations and Navier-Stokes equations of the boundary theory. The bulk theory also fixes the transport coefficients up to second order [143, 145, 148, 149]. Analogous results were also obtained for superfluids [150–152]. This "fluid/gravity" correspondence leads to a novel perspective on the membrane paradigm: we are

[17] In the case of a superfluid (i.e. when the $U(1)$ is spontaneously broken), one can use the two-component formalism [141], which also includes the superfluid density n_s and the superfluid velocity v_s (which is the gradient of the Goldstone boson), subject to the Josephson condition.

instructed to think of the physical fluid membrane as living on the boundary of the spacetime [153–155]. See also [156–160] for more work connecting gravity and fluid dynamics.

Let us comment in passing that even though hydrodynamics is an old and well studied subject, its derivation as an effective theory for generic QFTs in the spirit of Wilsonian renormalisation is still lacking. Only recently there has been some interesting progress in this direction, see [161–164].

Finally, we can also demand the existence of an entropy current J_S^μ such that the second law of thermodynamics would take the form $\nabla_\mu J_S^\mu \geq 0$, and this imposes constraints on the transport coefficients (for example $\eta, \zeta_b \geq 0$) [152, 165, 166]. The entropy current is quite intriguing since it is not defined in the underlying QFT, see [167] for a holographic definition using the fluid/gravity correspondence and [168–170] for some progress in understanding its origins from the abovementioned hydrodynamic effective actions.

Quasinormal modes

Black hole quasinormal modes are linearized perturbations of black hole backgrounds obeying infalling conditions at the horizon and appropriate asymptotic boundary conditions, depending on whether the solution is asymptotically flat, dS or AdS (see the reviews [171, 172]). They are of astrophysical interest, since they describe the long period of damped oscillations after perturbing a black hole. In case the black hole is dynamically unstable, there will also be exponentially growing quasinormal modes driving the system out of the perturbative regime, see also subsection 1.2.3 and chapter 3.

The discussion in subsection 1.1.3 suggests an elegant interpretation of quasinormal modes with vanishing non-normalizable boundary conditions in asymptotically AdS black hole backgrounds. By observing equation (1.1.46), we see that if we impose that the boundary sources $\delta\varphi_{B(0)}$ induced by the perturbations $\delta\varphi_B$ vanish, then the retarded Green's function G_{AB} will blow up. In other words, quasinormal modes correspond to the location of the poles of the retarded Green's functions of the dual field theory. This profound connection was conjectured and formulated in a series of papers, see [89, 173–176]. One can thus argue that quasinormal modes are the closest analogues of quasiparticles in strongly coupled holographic field theories.

Generally, it is difficult to find quasinormal modes, even though a variety of analytic and numerical techniques have been developed [171, 173, 177, 178]. However, in simple, relativistically invariant cases they can be found analytically, and in the zero wavelength $k \to 0$ limit they typically lie in the "christmas-tree" formation in the complex ω plane [175, 177–181]. As the temperature goes to zero $T \to 0$, these poles coalesce to form branch cuts.

The modes which approach the origin $\omega \to 0$ as $k \to 0$ correspond precisely to the modes that we can study from hydrodynamics. They are much easier to obtain, even analytically, than the rest of the "gapped" modes. This was done in $\mathrm{AdS}_5 \times S^5$ and M-theory in the early days of applied AdS/CFT [88, 89, 182, 183], and there have been many subsequent generalisations, see for example [85, 149, 176, 184–194] and the short reviews [195, 196]. This allowed the determination of various transport coefficients of the strongly coupled $\mathcal{N} = 4$ plasma independently of the fluid/gravity correspondence.

Arguably the most influential result was the computation of the shear viscosity (which was also obtained using other methods in [197, 198] and recently in [199]). It was shown that

$$\frac{\eta}{s} = \frac{1}{4\pi}\,, \tag{1.2.5}$$

where s is the entropy density. Remarkably, this value is very close to the experimentally determined value for the quark-gluon plasma [200], in contrast to the result at weak coupling $\eta/s \gg 1$, and this is considered to be one of the most successful "predictions" of AdS/CFT for applications to QCD [201]. The fact that quantum corrections are positive, together with the universality of (1.2.5) [202–204], led to the famous conjecture that $1/4\pi$ is a lower bound for η/s for every fluid in nature (the "KSS bound") [185, 198]. However, counterexamples have been found in higher curvature bulk theories [205–207] and in anisotropic backgrounds [208, 209][18], see [211] for a review of the relevant developments.

1.2.3 Symmetry breaking

Most of the progress in AdS/CMT that we reviewed so far was done in models which preserve translational symmetry. However, real-life condensed matter systems typically break translations explicitly through a background lattice structure and through impurities or quenched disorder [110], or they break translations spontaneously by the formation of charge or spin density waves [212, 213]. In order to obtain more realistic transport properties such as finite DC conductivities (see subsection 1.2.4), it is then desirable to study holographic models which break translational invariance (while preserving only a discrete subgroup to model the periodicity of the lattice).

Recall that in QFT, symmetries of the theory can be broken either spontaneously or explicitly. Explicit breaking takes place when we introduce sources which do not preserve the symmetry of the theory. For example, translations are broken

[18] However, the interpretation of the shear viscosity is more subtle when there is anisotropy [210].

explicitly if we introduce a space dependent source $J(x)$ coupling to the operator $\mathcal{O}$. Spontaneous breaking takes place when a state does not preserve the symmetry of the theory, or in other words when an operator acquires a VEV which does not preserve the symmetry. For example, translations are broken spontaneously if the operator $\mathcal{O}$ has a spatially dependent VEV $\langle\, \mathcal{O}(x)\, \rangle$.

Explicit breaking of translational invariance

Within the framework of holography it is conceptually simple to study explicit breaking of translational invariance: one needs to consider a background in which the leading boundary behavior of the fields is not homogeneous. However, this usually involves solving second PDEs in at least two variables, so we need to rely on numerical methods [214, 215] to construct the background. The first significant results were obtained by [216], in which the optical conductivity was also computed in the backreacted geometry sourced by a periodic scalar field. Various subsequent generalisations include [217–220].

Backgrounds which break translations explicitly are called "holographic lattices". They have led to a rich variety of ground states, moving beyond the classification mentioned in subsection 1.1.3. Examining the IR behavior, the lattice can be relevant or irrelevant with respect to an IR which preserves translations. In the latter case, the horizon restores translational symmetry and one can study transport by using perturbative techniques, while in the former case the backreaction of the lattice leads to novel spatially modulated ground states. By varying the lattice strength, metal-insulator transitions have been observed in various settings [221–228].

As already mentioned, there are a few QFT techniques which allow us to study transport in strongly coupled systems without translations. There has been progress in the weak momentum relaxation regime, in which momentum is almost conserved, i.e. it decays over a large timescale τ_R. Perturbation theory in τ_R^{-1}, sometimes in conjuction with the memory matrix formalism or hydrodynamics has been used over the past few years, and connections to holographic systems have been explored [117, 132, 135, 229–237]. Alternatively, one could modify the Ward identities by adding a "phenomenological" momentum relaxation term, as in [132, 238]. Another fruitful approach is the introduction of disorder by taking the momentum relaxing sources to belong in some statistical ensemble. RG flows triggered by disorder have been studied within holography [230, 239–242], and also purely in field theory [243–245].

Spontaneous breaking of translational invariance

The mechanism for breaking translations spontaneously is the same as the one proposed in the seminal papers [246, 247] in the case of spontaneous breaking of a bulk $U(1)$ gauge symmetry[19]. Using a charged scalar ψ in the bulk, they showed that the "normal" or "disordered" phase with $\psi = 0$ was not thermodynamically preferred below a critical temperature T_c. Instead, a solution with a normalisable mode for ψ (the "broken" or "ordered" phase) had lower free energy below T_c, and so the dual scalar operator $\mathcal{O}$ acquired a non-zero VEV $\langle \mathcal{O} \rangle$, breaking the bulk $U(1)$ by the Higgs mechanism. Such solutions are allowed because black holes in AdS spacetimes evade the usual no-hair theorems [248].

The construction of holographic superconductors has been extended in various ways [249–253]. The addition of more parameters in the system, such as a double trace deformation [254, 255] or magnetic fields [256], leads to interesting phase diagrams.

A good indication for the existence of a broken phase at low temperatures is the identification of an unstable perturbative mode in the near horizon region. Heuristically, this instability comes from the fact that, as mentioned in subsection 1.1.3, at low enough temperatures the near horizon region approaches the form of $\mathrm{AdS}_2 \times \mathbb{R}^d$, see equation (1.1.41). Let us consider for simplicity a probe neutral[20] scalar φ with mass m in the near horizon region. It satisfies the wave equation

$$\left[\Box_{AdS_2} - m^2_{eff}\right] \varphi = 0\,, \qquad m^2_{eff} = m^2 + k^2 \,\frac{d(d+1)e^2}{(d-1)^2\mu^2}\,, \tag{1.2.6}$$

where m_{eff} is the effective mass of φ which depends on the momentum $k^2 \equiv k^i k^i$ in $\mathbb{R}^d$ (if the scalar was charged, m_{eff} would also receive negative contributions from the gauge field). It is then possible that, for some range of momenta k^2, the effective mass m_{eff} of φ violates the near horizon BF bound (1.1.19), i.e. $m^2_{eff} L_2^2 < -1/4$, even though m satisfies the $\mathrm{AdS_{d+2}}$ BF bound. We can see that this leads to a dynamical instability by recalling that the scaling dimension Δ_{IR} of the IR CFT operator $\mathcal{O}$ dual to φ is given by

$$\Delta_{IR} = \frac{1}{2} + \sqrt{m^2_{eff} L_2^2 + \frac{1}{4}}\,. \tag{1.2.7}$$

A violation of the AdS_2 BF bound results in a complex scaling dimension Δ_{IR} and this leads to the low energy retarded Green's function containing poles in the upper half complex frequency plane [71]. This signifies a dynamical instability.

The above analysis is valid in the zero temperature limit $T \to 0$. By raising T,

[19] In the boundary theory this corresponds to breaking a global $U(1)$, so this mechanism describes superfluids rather than superconductors. However, this distinction will not be important in the following.

[20] This will not lead to a broken $U(1)$ gauge symmetry, but it will still describe the holographic mechanism for the condensation of a boundary operator.

the black hole "swallows" the near horizon instability region[21]. Thus, there generally exist unstable modes with a range of momenta k, bounded by a bell-shaped curve in the $T-k$ plane. The maximum of this curve is expected to correspond to the critical temperature T_c and the periodicity k_c of the backreacted broken phase. From equation (1.2.6) we can see that in this simple example $k_c = 0$. However we could also have mixing of various modes, in which case equation (1.2.6) generalises to a matrix equation. The eigenvalues of the mass matrix depend on the parameters of the theory and of the background, and by varying the latter accordingly we may be able to get instability curves with $k_c \neq 0$. Then, the broken phases are expected to exhibit spontaneous spatial modulation [259–261]. Such instabilities have been observed in various settings, including cases with magnetic fields [256, 262–277]. Usually one needs to consider bulk actions with topological Chern-Simons or theta terms, but even the EMD action (1.1.34) contains instabilities for some forms of the functions $V(\phi)$ and $Z(\phi)$. Note that most of these are first order phase transitions.

Before closing this subsection, it is worth noting that there is an easier way to obtain backgrounds which break translations explicitly or spontaneously using helical lattices [261] or the Q-lattice construction of [278] (or the slightly simpler linear axions models [279]). Specifically, in the latter case, a bulk global $U(1)$ symmetry is exploited in order to break translations only in the scalar sector; the metric and the gauge field remain homogeneous[22]. This leads to ODEs instead of PDEs, which are much easier to solve numerically. Finally, let us mention that an alternative way to break translations is by using massive gravity in the bulk [280–284].

1.2.4 Thermoelectric conductivities

Let us now give some more details about the thermoelectric conductivities (1.2.1), which will play an instrumental role throughout this thesis.

All of the QFT techniques outlined previously lead to the following form for the low frequency electric conductivity [132]:

$$\sigma(\omega) = \sigma_Q + \frac{\rho^2}{\epsilon + P} \frac{1}{\tau_R^{-1} - i\,\omega} + \dots\,, \tag{1.2.8}$$

[21] Note however that there are can also be unstable modes which not localised near the horizon, see [257, 258] and the discussion in chapter 3.

[22] As we mentioned in footnote 7, the bulk quantum gravity theory should not have global symmetries. However, here we are taking the classical limit in the bulk, so it is possible that the $U(1)$ symmetry is broken by quantum effects. In a more phenomenological spirit, we can think of our theory as a subsector of a larger theory which does not respect this symmetry [278]. This point will not concern us further, but we note that it is an interesting open question whether it has any meaning in the dual boundary theory.

where τ_R is the momentum relaxation rate which is larger than any other scale, and the dots denote corrections in ω and τ_R^{-1}. The second term has the well known kinetic theory Drude form, leading to the Drude peak at low frequencies. In contrast, the Drude peak is absent in "incoherent" metals, i.e. strange metals with strong momentum relaxation.

In general, it is a hard problem in condensed matter physics to calculate the AC conductivity in systems without quasiparticles beyond the leading order Drude form. However, using holography, the Drude peak (1.2.8) was reproduced and corrections were calculated in simple systems [80, 133, 134, 223, 238, 278, 285, 286]. We will not discuss the AC conductivities further, but let us mention that historically, [287] sparked the community's interest by proving the remarkable result that, in the EMD theory (1.1.34) in 4 bulk dimensions, at finite temperature, the electric AC conductivity is independent of ω and T:

$$\sigma = \frac{1}{e^2}\,. \tag{1.2.9}$$

This comes from electromagnetic duality in the bulk, which is related to particle-vortex duality at the boundary. Later, [288] showed that higher derivative Weyl curvature corrections, which spoil the duality, can give particle-like peak or vortex-like dip, depending on the sign of the coupling.

We end the introduction by discussing the calculation of the thermoelectric DC conductivities from black hole horizons. The idea originated from [85], which considered translationally invariant, neutral systems and showed that the parts of the current and the source relevant to the DC calculation are not renormalised as we move from the boundary into the bulk. Thus, the electric DC conductivity σ_{DC} was obtained as a horizon expression. Later, [222, 224, 289] noticed that even in the Q-lattices, the thermoelectric conductivities (which are all finite) are given by horizon expressions. The generalisation to systems with spatial modulation in one dimension was done in [285], and finally the general formalism was laid out in [45, 290].

Let us review the main elements of this prescription; a sketch with explicit expressions will also be given in appendix 3.C. The setup involves the EMD action and a general ansatz describing a static, periodic, inhomogeneous black hole background, see figure 1.1. One then considers a perturbation with a linear in time part which introduces an electric field E_i and a thermal gradient ζ_i at the boundary, where i runs over the spatial directions. In the bulk, one can define an electric current J^i and a 1-form Q^i[23]. Expressing them in terms of the perturbations, it can be shown

[23] In [291] the "bulk heat current" was obtained for general bulk theories by doing a timelike Kaluza-Klein reduction. An alternative viewpoint is to consider it as the Noether current associated

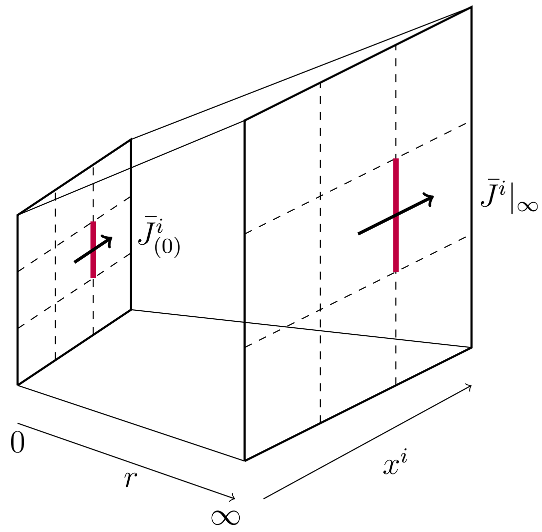

Figure 1.1: The setup for the holographic calculation of the DC conductivities in an inhomogeneous black hole background with discrete lattice symmetry. The bulk radial coordinate goes from 0 at the horizon to ∞ at the AdS boundary and the spatial coordinates x^i are periodic. The boundary electric current flux $\bar{J}^i|_\infty$ in the i-direction is equal to the horizon electric current flux $\bar{J}^i_{(0)}$ in the i-direction, and similarly for the heat current.

that they give the electric and heat currents, $J^i|_\infty$ and $Q^i|_\infty$, of the dual QFT. On the horizon they reduce to what we define as the horizon electric and heat currents, $J^i_{(0)}$ and $Q^i_{(0)}$, see (3.C.2).

We can then evaluate the constraints (1.1.28) at the black hole horizon[24]. The result is a closed system of Navier-Stokes equations (3.C.1), in terms of a subsector of the bulk perturbations evaluated on the horizon and the DC sources E_i and ζ_i. By solving this system we obtain the horizon currents $J^i_{(0)}\,,Q^i_{(0)}$, and thus the current fluxes $\bar{J}^i_{(0)} \equiv \oint J^i_{(0)}\,,\bar{Q}^i_{(0)} \equiv \oint Q^i_{(0)}$ (by $\oint$ we denote the integral over a lattice cell), in terms of E_i and ζ_i. By virtue of the bulk equations of motion, it can be shown that $\bar{J}^i_{(0)}\,,\bar{Q}^i_{(0)}$ are equal to the boundary current fluxes $\bar{J}^i|_\infty\,,\bar{Q}^i|_\infty$, even though the local currents generally disagree. The fluxes are precisely what is relevant for transport and so, from the definition (1.2.1) we can calculate the thermoelectric DC conductivities.

The previous results for the conductivities from the horizon were then understood as special cases where the Navier-Stokes equations can be solved explicitly. From this general result one can also obtain the weak momentum relaxation results by considering a weakly curved background. This prescription was extended

with the Killing vector ∂_t of the static background, within the Wald formalism [67, 68], as was done in [199, 292].

[24]See also the comments in footnote 9.

to backgrounds with magnetic fields [293] and to higher derivative bulk theories [291]. Also, [294] considered the thermodynamic limit, in which the periodicity of the background is small compared to the temperature. Up to the relevant order, the horizon is effectively pushed closer to the boundary and thus even the local horizon and boundary currents coincide. This then reproduces the results of the fluid/gravity correspondence.

Chapter 2

Diffusion in inhomogeneous media

This chapter is a reproduction of [1], written in collaboration with Aristomenis Donos and Jerome Gauntlett.

In this paper we study transport of conserved charges in inhomogeneous QFTs with a discrete lattice symmetry. In view of applications to incoherent metals, we make no reference to the strength of momentum relaxation or the strength of the coupling. In this case, we expect that transport is dominated by the diffusion of conserved charges. It is well known that the Einstein relation (see (2.1.1) below), which relates the diffusion constant to the DC conductivity and the static susceptibility has a wide range of applicability, from Brownian motion and lattice systems [110] to neutral homogeneous QFTs [131] and simple holographic field theories [85, 185]. In this paper we addressed the question of whether and how it can also be extended to spatially inhomogeneous quantum systems.

In section 2.2 we consider the retarded two point functions of the charges and the associated currents. The discrete lattice symmetry implies that the correlators depend on a continuous and a discrete wavevector. The continuous one can be arbitrarily small and can describe "hydrodynamic" phenomena, on scales much larger than the lattice scale. The non-trivial discrete wavevectors are parametrically larger, so we focus on the "zero lattice modes" of the correlators. Indeed, we find that these quantities are precisely what is relevant to transport, and much of the technology for spatially homogeneous systems can also be developed for this case. We are thus able to show the existence of a set of diffusive modes associated with the charges. The dispersion relations of these modes are related to the eigenvalues of a specific matrix constructed from the DC conductivities and certain thermodynamic susceptibilities, thus obtaining generalised Einstein relations. In the course of the proof, we assumed that a quantity constructed from the retarded two point functions (2.2.43) does not have poles at the origin of the complex frequency plane. This technical assumption is well motivated physically, since we expect that generic QFTs do not contain “accidentally” long-lived modes, not coming from a conservation law. We also assumed that the DC conductivities are finite, which means that momentum has to relax. Indeed, in the opposite case there is only one diffusive mode [134].

In section 2.3, we illustrate these general results in the specific context of relativistic hydrodynamics where translation invariance is broken using spatially inhomogeneous and periodic deformations of the stress tensor and the conserved $U(1)$ currents. Equivalently, this corresponds to considering hydrodynamics on a curved manifold, with a spatially periodic metric and chemical potential. By performing a specific long-wavelength expansion, we are able to explicitly construct the heat and charge diffusive modes, and obtain their dispersion relations, confirming the results of section 2.2. We also make contact with [231] by explaining the derivation of the “reduced” hydrodynamic formalism without conserved momentum, used in order to obtain the generalised Einstein relations.

2.1 Introduction

Motivated by various strongly correlated states of matter seen in Nature, there has been a significant effort devoted to obtaining a deeper theoretical understanding of thermoelectric transport. It has long been appreciated that it is necessary to work within a framework in which momentum is not conserved. Indeed, for a translationally invariant system in which momentum is exactly conserved, the AC thermal response necessarily contains a delta function at zero frequency leading to a non-physical infinite DC thermal conductivity. Thus, one is interested in setups in which translation symmetry is explicitly broken.

In this paper we will present some general results for thermoelectric transport in inhomogeneous systems. More precisely, we will consider general quantum systems, with one or more conserved currents, with a discrete, spatial lattice symmetry. This could describe, for example, a quantum field theory in which translation invariance has been explicitly broken by deforming the theory with operators which have a periodic dependence on the spatial coordinates.

Of central interest are the retarded Green’s functions for the current-current correlators $G_{JJ}(t,\mathbf{x};t'\mathbf{x}')$. At the level of linear response these determine how the currents respond after perturbing the system by a current source. Time translation invariance implies that these Green’s functions only depend on $t-t'$ which allows us to Fourier transform and obtain $G_{JJ}(\omega,\mathbf{x},\mathbf{x}')$. In a translationally invariant setting the Green’s functions would also only depend on $\mathbf{x}'-\mathbf{x}$ and a Fourier transform leads to a correlator depending on ω and a single wave-vector $\mathbf{k}$. When translations are broken, this is no longer possible but a discrete lattice symmetry allows us to define an infinite discrete set of correlators $G_{JJ}^{(\{n_i\})}(\omega,\mathbf{k})$, where $\{n_i\}$ are a set of integers. We will be particularly interested in studying the correlator $G_{JJ}(\omega,\mathbf{k}) \equiv G_{JJ}^{(\{0\})}(\omega,\mathbf{k})$. Indeed this correlator, which satisfies a simple positivity condition, captures the transport properties of the system at late times and for wavelengths much longer than the scale of the lattice, and thus we might call $G_{JJ}(\omega,\mathbf{k})$ a 'hydrodynamic-mode correlator'.

By generalising similar computations presented in [131] in the translationally invariant setting, we will show that when the thermoelectric DC conductivity is finite, subject to some analyticity assumptions, there is necessarily a diffusion pole in the hydrodynamic mode correlator for $\omega, \varepsilon\mathbf{k} \to 0$. If the system has just a single conserved current then we explain precisely when we get a dispersion relation for the diffusion pole of the form

$$\omega = -i\varepsilon^2 D(\mathbf{k}) + \dots\,, \qquad D(\mathbf{k}) = [\sigma_{DC}^{ij} k_i k_j]\chi(\mathbf{0})^{-1}\,, \tag{2.1.1}$$

where σ_{DC}^{ij} is the DC conductivity and $\chi(\mathbf{k})$ is the charge susceptibility. This is our first Einstein relation for inhomogeneous media.

When there are additional conserved currents, there will be additional diffusion modes when the associated DC conductivities are finite. We analyse the dispersion relations for the diffusion modes and show how they can be obtained from the eigenvalues of a specific 'generalised diffusion matrix' that is constructed from the DC conductivities and various thermodynamic susceptibilities. We emphasise that, generically, the dispersion relations for the diffusion modes are not of the form (2.1.1) and hence we refer to our result concerning the dispersion relation as a 'generalised Einstein relation'. This feature of diffusion modes was also emphasised in [231] within a specific hydrodynamic setting, which we will return to later.

These results concerning hydrodynamic modes of the Green's functions are very general. However, motivated by recent experimental progress [295–297], there has been considerable theoretical work using hydrodynamics to study thermoelectric transport [134, 135, 231, 233, 234, 237, 238, 298–304] and it is therefore of interest to see how our general results on diffusion manifest themselves in this particular context. More specifically, we will study this within the context of relativistic hydrodynamics, describing the hydrodynamic limit of a relativistic quantum field theory.

Within this hydrodynamic framework, we first need to consider how momentum dissipation is to be incorporated. A standard approach is to modify, by hand, the hydrodynamic equations of motion, i.e. the Ward identities of the underlying field theory, by a phenomenological term that incorporates momentum dissipation (e.g. [132, 142]). An alternative and more controlled approach is to maintain the Ward identities, which are fundamental properties of the field theory, but to consider the field theory to be deformed by spatially dependent sources. In this spirit, the hydrodynamic limit of a class of field theories which have been deformed by certain scalar operators was analysed in [234]. Subsequently, the universal class of deformations which involve adding spatially dependent sources for the stress tensor were studied in [300]. Since the stress tensor of the field theory couples to the spacetime metric, the deformations studied in [300] are equivalent to studying the hydrodynamic limit of the quantum field theory on a curved spacetime manifold. The spacetime metric is taken to have a time-like Killing vector in order to discuss thermal equilibrium. Then, while spatial momentum will, generically, no longer be conserved, energy still will be. It may be possible to experimentally realise the deformations studied in [300] in real materials, such as strained graphene [305–307].

In this paper we extend the analysis of [300] to cover relativistic quantum field theories which have a conserved $U(1)$ symmetry[1]. As in [300] we can consider the field theory to live on a static, curved manifold. Although not necessary, it will be convenient to take the manifold to have planar topology and with a metric that is periodic in the spatial directions. Within the hydrodynamic framework we will also consider deformations that are associated with spatially dependent sources for the $U(1)$ symmetry. This is particularly interesting since it corresponds to allowing for spatially dependent chemical potential or, equivalently, spatially dependent charge density. One can anticipate that our results will be useful for understanding thermoelectric transport in real systems, such as charged puddles, with or without strain, in graphene [308–311] as also discussed in [304].

As an application of our formalism, we show how to construct long-wavelength, late-time hydrodynamic modes that are associated with diffusion of both energy and electric charge. We derive the dispersion relation for these modes and explicitly obtain the generalised Einstein relations. It is worth noting that this result is independent of the precise transport coefficients that enter the constitutive relations in the conserved currents. We also note that a derivation of an Einstein relation for the diffusion of electric charge in the context of hydrodynamics with vanishing local charge density in one spatial direction was carried out in appendix A of [299] and this is consistent with our more general analysis here.

[1]While writing up this work, ref. [304] appeared which also generalises [300] to include a conserved $U(1)$ charge and independently derived the hydrodynamic equations (2.3.23), for the special case of no time dependence and for curved manifolds with a unit norm timelike Killing vector (i.e. $f = 1$).

2.2 Green's function perspective

We begin our discussion with a general quantum system with a time independent Hamiltonian H. We assume that there is a lattice symmetry group which acts on the d spatial coordinates via $\mathbf{x} \to \mathbf{x} + \mathbf{L}_j$ and $U_{\mathbf{L}_j}^{-1} A(t, \mathbf{x}) U_{\mathbf{L}_j} = A(t, \mathbf{x} + \mathbf{L}_j)$, where $A(t, \mathbf{x})$ is an arbitrary local operator. We assume that the Hamiltonian is invariant under this symmetry and hence $U_{\mathbf{L}_j}^{-1} H U_{\mathbf{L}_j} = H$. We will also consider the system to be at finite temperature T.

As usual, for two local operators $A(t, \mathbf{x})$, $B(t, \mathbf{x})$, the retarded two point functions are defined through

$$G_{AB}(t, \mathbf{x}; t', \mathbf{x}') = -i\theta(t - t') \langle [A(t, \mathbf{x}), B(t', \mathbf{x}')] \rangle \,, \tag{2.2.1}$$

with $\langle A(t, \mathbf{x}) \rangle = \mathrm{Tr}\,(\rho A(t, \mathbf{x}))$, where $\rho = e^{-\beta H} / \,\mathrm{Tr}(e^{-\beta H})$ and $\beta = 1/T$. Using the fact that $A(t, \mathbf{x}) = e^{itH} A(0, \mathbf{x}) e^{-itH}$ and the lattice symmetry of H, we see that the two point functions will satisfy

$$G_{AB}(t, \mathbf{x}; t', \mathbf{x}') = G_{AB}(t - t', \mathbf{x}; 0, \mathbf{x}') \,, \tag{2.2.2}$$

$$G_{AB}(t, \mathbf{x} + \mathbf{L}_j; t', \mathbf{x}' + \mathbf{L}_j) = G_{AB}(t, \mathbf{x}; t', \mathbf{x}') \,. \tag{2.2.3}$$

The symmetry (2.2.2) allows us to define a function with three arguments through $G_{AB}(t - t', \mathbf{x}, \mathbf{x}') \equiv G_{AB}(t, \mathbf{x}; t', \mathbf{x}')$.

We next recall that if we introduce a perturbative source term in the Hamiltonian via

$$\delta H(t) = \int d\mathbf{x}\, \delta h_B(t, \mathbf{x}) B(t, \mathbf{x}) \,, \tag{2.2.4}$$

then at the level of linear response, the change in the expectation values of an arbitrary operator A is given by

$$\delta\langle A \rangle(t, \mathbf{x}) = \int dt' d\mathbf{x}'\, G_{AB}(t - t', \mathbf{x}, \mathbf{x}') \delta h_B(t', \mathbf{x}') \,. \tag{2.2.5}$$

We note that the source, and hence the response, need not be a periodic function of the spatial coordinates and indeed this will be case of most interest in the following.

To proceed we Fourier transform the Green's function on all arguments and define

$$G_{AB}(\omega, \mathbf{k}, \mathbf{k}') \equiv \int dt d\mathbf{x} d\mathbf{x}'\, e^{i\omega t - i\mathbf{k}\mathbf{x} + i\,\mathbf{k}'\mathbf{x}'}\, G_{AB}(t, \mathbf{x}, \mathbf{x}') \,. \tag{2.2.6}$$

The discrete symmetry (2.2.3) implies that we can perform a crystallographic type of decomposition to obtain

$$G_{AB}(\omega, \mathbf{k}, \mathbf{k}') = \sum_{\{n_j\}} G_{AB}^{(\{n_j\})}(\omega, \mathbf{k}')\, \delta(\mathbf{k} - \mathbf{k}' - n_j\, \mathbf{k}_L^j) \,, \tag{2.2.7}$$

where $\mathbf{k}_L^j$ are the reciprocal lattice vectors satisfying $\mathbf{k}_L^i \cdot \mathbf{L}^j = 2\pi\delta^{ij}$ and $\{n_j\}$ are sets of integers. To see this, we simply notice that if we define the function

$$G_{AB}(\omega, \mathbf{x}, \mathbf{k}') \equiv \int dt d\mathbf{x}' \, e^{i\omega t + i\,\mathbf{k}'\mathbf{x}'} \, G_{AB}(t, \mathbf{x}, \mathbf{x}') \,, \tag{2.2.8}$$

then the real space lattice symmetry (2.2.3) implies the periodicity condition

$$G_{AB}(\omega, \mathbf{x} + \mathbf{L}_j, \mathbf{k}') = e^{i\,\mathbf{k}'\mathbf{L}_j} \, G_{AB}(\omega, \mathbf{x}, \mathbf{k}') \,, \tag{2.2.9}$$

and hence we can deduce that $e^{-i\,\mathbf{k}'\mathbf{x}} \, G_{BB'}(\omega, \mathbf{x}, \mathbf{k}')$ is periodic as a function of $\mathbf{x}$. This lets us write it as a discrete Fourier series, expressing

$$G_{AB}(\omega, \mathbf{x}, \mathbf{k}') = \frac{1}{(2\pi)^d} \, e^{i\,\mathbf{k}'\mathbf{x}} \sum_{\{n_j\}} e^{in_j \mathbf{k}_L^j \mathbf{x}} G_{AB}^{(\{n_j\})}(\omega, \mathbf{k}') \,, \tag{2.2.10}$$

and (2.2.7) follows.

In the sequel, we will be particularly interested in the zero modes, $G_{AB}(\omega, \mathbf{k}) \equiv G_{AB}^{(\{0\})}(\omega, \mathbf{k})$. These can easily be obtained by taking average spatial integrals over a period of periodic functions. If we define $\oint \equiv (\prod L_i)^{-1} \int_{\{0\}}^{\{\mathbf{L}_i\}} d\mathbf{x}$ then we have

$$G_{AB}(\omega, \mathbf{k}) \equiv G_{AB}^{(\{0\})}(\omega, \mathbf{k}) = \oint d\mathbf{x} \int d\mathbf{x}' \, G_{AB}(\omega, \mathbf{x}, \mathbf{x}') e^{i\mathbf{k}(\mathbf{x}'-\mathbf{x})} \,. \tag{2.2.11}$$

From (2.2.6) we can also write

$$G_{AB}(\omega, \mathbf{k}) = (N \prod_i L_i)^{-1} G_{AB}(\omega, \mathbf{k}, \mathbf{k}) \,, \tag{2.2.12}$$

where N is the total number of spatial periods in the system.

We next examine the positivity of the spectral weight of our operators. Working in the interaction picture, the system absorbs energy at rate

$$\frac{d}{dt} W(t) = \int d\mathbf{x} \, \delta\langle B\rangle(t, \mathbf{x}) \, \frac{d}{dt} \delta h_B(t, \mathbf{x}) \,, \tag{2.2.13}$$

where a summation over B is understood. Introducing the notation

$$\delta h_B(t, \mathbf{x}) = \frac{1}{(2\pi)^{d+1}} \int d\omega d\mathbf{k} \, \delta h_B(\omega, \mathbf{k}) \, e^{-i\omega t + i\mathbf{k}\mathbf{x}} \,, \tag{2.2.14}$$

we can show that the total energy absorbed by the system is

$$\Delta W = -\frac{1}{(2\pi)^{2d+1}} \int d\omega d\,\mathbf{k} d\mathbf{k}' \delta h_B^*(\omega, \mathbf{k}) \omega [\mathrm{Im} G]_{BB'}(\omega, \mathbf{k}, \mathbf{k}') \delta h_{B'}(\omega, \mathbf{k}') \,, \tag{2.2.15}$$

where $[\mathrm{Im}G]_{AB}(\omega, \mathbf{k}, \mathbf{k}') \equiv \frac{1}{2i}[G_{AB}(\omega, \mathbf{k}, \mathbf{k}') - G_{BA}^*(\omega, \mathbf{k}', \mathbf{k})]$. To get to the last line we used $G_{AB}(\omega, \mathbf{k}, \mathbf{k}') = G_{AB}(-\omega, -\mathbf{k}, -\mathbf{k}')^*$ (for real frequencies and wavevectors), which follows from the reality of $G_{AB}(t, \mathbf{x}, \mathbf{x}')$. Since $\delta h_B(\omega, \mathbf{k})$ are arbitrary we deduce that $-\omega[\mathrm{Im}G]_{AB}(\omega,\mathbf{k},\mathbf{k}')$ is a positive semi-definite matrix, with matrix indices including both the operator labels as well as the wavevectors. Since the block diagonal elements of a positive semi-definite matrix are positive semi-definite, using (2.2.12) we can conclude that the zero modes $-\omega ImG_{AB}(\omega,\mathbf{k})$ are positive semi-definite. In particular we have

$$-\omega ImG_{AA}(\omega,\mathbf{k})\geq 0\,, \tag{2.2.16}$$

with no sum on A. The positive semi-definite aspect of $-\omega[\mathrm{Im}G]_{AB}(\omega,\mathbf{k},\mathbf{k}')$ also gives rise to additional conditions for the $G_{AB}^{(\{n_j\})}(\omega,\mathbf{k})$, with $\{n_j\}\neq\{0\}$.

To conclude this subsection we examine how the Green's functions behave under time reversal invariance. For simplicity we will assume that the periodic system is invariant under time reversal. Recall that this acts on local operators according to $T\,A(t,\mathbf{x})\,T^{-1}=\epsilon_A A(-t,\mathbf{x})$, where $\epsilon_A=\pm 1$. Since T is an anti-unitary operator we can deduce that $G_{AB}(t,\mathbf{x},\mathbf{x}')=\epsilon_A\epsilon_B\,G_{BA}(t,\mathbf{x}',\mathbf{x})$. Thus, we have $G_{AB}(\omega,\mathbf{k},\mathbf{k}')=\epsilon_A\epsilon_B\,G_{BA}(\omega,-\mathbf{k}',-\mathbf{k})$ and hence

$$G_{AB}^{(\{n_j\})}(\omega,\mathbf{k})=\epsilon_A\,\epsilon_B\,G_{BA}^{(\{n_j\})}(\omega,-\mathbf{k}-n_l\mathbf{k}_L^l)\,. \tag{2.2.17}$$

Returning to the linear response given in (2.2.5), after taking suitable Fourier transforms we can write

$$\delta\langle\,A\,\rangle(\omega,\mathbf{x})=\frac{1}{(2\pi)^{2d}}\int d\mathbf{k}\,\sum_{\{n_j\}}e^{i(\mathbf{k}+n_j\,\mathbf{k}_L^j)\mathbf{x}}G_{AB}^{(\{n_j\})}(\omega,\mathbf{k})\delta h_B(\omega,\mathbf{k})\,. \tag{2.2.18}$$

If we consider a source which contains a single spatial Fourier mode $\delta h_B(t,\mathbf{x})=e^{i\mathbf{k}_s\mathbf{x}}\delta h_B(t)$, then we have

$$\begin{aligned}\delta\langle\,A\,\rangle(\omega,\mathbf{x})&=e^{i\mathbf{k}_s\mathbf{x}}\sum_{\{n_j\}}\frac{1}{(2\pi)^d}e^{in_j\,\mathbf{k}_L^j\mathbf{x}}G_{AB}^{(\{n_j\})}(\omega,\mathbf{k}_s)\delta h_B(\omega)\,,\\&\equiv e^{i\mathbf{k}_s\mathbf{x}}\sum_{\{n_j\}}e^{in_j\mathbf{k}_L^j\mathbf{x}}\,\delta\langle\,A\,\rangle^{(\{n_j\})}(\omega,\mathbf{k})\,.\end{aligned} \tag{2.2.19}$$

Notice, in particular, that the zero mode in the summation is fixed by the zero mode of the Green's function: $\delta\langle\,A\,\rangle^{(\{0\})}(\omega,\mathbf{k})=(2\pi)^{-d}G_{AB}(\omega,\mathbf{k}_s)\delta h_B(\omega)$.

In the next sub-sections we will take A and B to be components of conserved currents. In this context the zero-mode correlator $G_{AB}(\omega,\mathbf{k})$ captures transport of the associated hydrodynamic modes and hence one can call it a 'hydrodynamic-mode correlator'.

2.2.1 Einstein relation for a single current

We now consider the operator A to be a current density[2] operator J^μ, which satisfies a continuity equation of the form $\partial_\mu J^\mu = 0$. From the definition (2.2.6) we have

$$-i\omega\, G_{J^t B}(\omega, \mathbf{k}, \mathbf{k}') + i\mathbf{k}_i G_{J^i B}(\omega, \mathbf{k}, \mathbf{k}') = 0\,, \tag{2.2.20}$$

for any operator B, whose equal time commutator with J^t vanishes. Using the crystallographic decomposition (2.2.7) in (2.2.20) we then have

$$-i\omega\, G^{(\{n_j\})}_{J^t B}(\omega, \mathbf{k}) + i\left(\mathbf{k} + n_j \mathbf{k}_L^j\right)_i G^{(\{n_j\})}_{J^i B}(\omega, \mathbf{k}) = 0\,. \tag{2.2.21}$$

We now[3] focus on the hydrodynamic-mode correlators with $\{n_j\} = 0$, which satisfy a positivity property discussed just above (2.2.16). Using (2.2.21) twice, we have

$$\begin{aligned} -i\omega\, G_{J^t J^t}(\omega, \mathbf{k}) + i\mathbf{k}_i G_{J^i J^t}(\omega, \mathbf{k}) =& 0\,, \\ -i\omega\, G_{J^t J^j}(\omega, \mathbf{k}) + i\mathbf{k}_i G_{J^i J^j}(\omega, \mathbf{k}) =& 0\,. \end{aligned} \tag{2.2.22}$$

We next consider the time reversal invariance conditions (2.2.17) with $\{n_j\} = 0$. Since $\epsilon_{J^t} = +1$ and $\epsilon_{J^i} = -1$, we obtain

$$\begin{aligned} G_{J^i J^t}(\omega, \mathbf{k}) &= -G_{J^t J^i}(\omega, -\mathbf{k})\,, \\ G_{J^i J^j}(\omega, \mathbf{k}) &= G_{J^j J^i}(\omega, -\mathbf{k})\,. \end{aligned} \tag{2.2.23}$$

Combing (2.2.23) with (2.2.22) we therefore have the key result

$$\frac{1}{i\omega}\mathbf{k}_i \mathbf{k}_j\, G_{J^i J^j}(\omega, \mathbf{k}) = -i\omega\, G_{J^t J^t}(\omega, \mathbf{k})\,. \tag{2.2.24}$$

In general, taking the $\omega \to 0$ limit of the correlator $G_{AB}(\omega, \mathbf{k})$ gives rise to a static, thermodynamic susceptibility. It will be useful to write

$$-\lim_{\omega \to 0 + i0} G_{J^t J^t}(\omega, \mathbf{k}) \equiv \chi(\mathbf{k})\,, \tag{2.2.25}$$

where $\chi(\mathbf{k})$ is a charge-charge susceptibility (the sign here is explained in appendix 2.A). Note that (2.2.24) implies

$$\lim_{\omega \to 0 + i0} \frac{1}{\omega^2}\mathbf{k}_i \mathbf{k}_j\, G_{J^i J^j}(\omega, \mathbf{k}) = -\chi(\mathbf{k})\,, \tag{2.2.26}$$

and in particular, the longitudinal part of the current-current susceptibility vanishes,

---

[2]In this section we find it convenient to work with current vector densities. In section 2.3 we will work with current vectors. We also note that as our analysis will focus on two-point functions of the current, we only require that the current to be conserved at the linearised level.

[3]We have also presented some more general results in appendix 2.A.

$\lim_{\omega\to 0+i0} \mathbf{k}_i\mathbf{k}_j G_{J^iJ^j}(\omega,\mathbf{k}) = 0$, provided that $\chi(\mathbf{k})$ is finite[4], which we will assume.

In order to focus on studying the response to long wavelength sources, it will be convenient to now rescale the wave-number $\mathbf{k}$ by ε and write (2.2.24) in the form

$$\frac{1}{i\omega}\mathbf{k}_i\mathbf{k}_j\, G_{J^iJ^j}(\omega,\varepsilon\,\mathbf{k}) = -\frac{i\omega}{\varepsilon^2}\, G_{J^tJ^t}(\omega,\varepsilon\mathbf{k})\,. \tag{2.2.27}$$

We next note that the AC conductivity matrix is defined by taking the following limit of the transport correlators

$$\sigma^{ij}(\omega) = -\lim_{\varepsilon\to 0}\frac{1}{i\omega}G_{J^iJ^j}(\omega,\varepsilon\,\mathbf{k})\,. \tag{2.2.28}$$

Notice from the discussion above (2.2.16) that the real part of $\sigma^{ij}(\omega)$ is a positive semi-definite matrix. In general, the AC conductivity is a finite quantity for $\omega\neq 0$. On the other hand, the DC conductivity, defined by $\sigma^{ij}_{DC}\equiv\lim_{\omega\to 0}\sigma(\omega)$, is not necessarily finite. For example, if the system is translationally invariant or if the breaking of translation invariance has arisen spontaneously, or more generally if there are Goldstone modes present, generically the DC conductivity will be infinite, or more precisely there will be a delta function on the AC conductivity at $\omega = 0$. By taking the limit $\varepsilon\to 0$ in (2.2.27) we have

$$\mathbf{k}_i\mathbf{k}_j\,\sigma^{ij}(\omega) = i\omega\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\, G_{J^tJ^t}(\omega,\varepsilon\mathbf{k})\,. \tag{2.2.29}$$

Thus when the *DC* conductivity is finite, the function $G_{J^tJ^t}(\omega,\varepsilon\mathbf{k})/\varepsilon^2$ must have a pole at $\omega = 0$ after taking the $\varepsilon\to 0$ limit. Note that (2.2.25) shows that before the limit is taken this pole is absent (provided that $\chi(\mathbf{k})$ is finite).

To make further progress, it is helpful to write

$$G_{J^tJ^t}(\omega,\varepsilon\mathbf{k})\chi(\varepsilon\mathbf{k})^{-1} = \frac{-N(\omega,\varepsilon\mathbf{k})}{-i\omega+N(\omega,\varepsilon\mathbf{k})}\,, \tag{2.2.30}$$

where we have defined the quantity

$$N(\omega,\varepsilon\mathbf{k}) = \frac{G_{J^tJ^t}(\omega,\varepsilon\mathbf{k})}{\frac{1}{i\omega}(G_{J^tJ^t}(\omega,\varepsilon\mathbf{k})+\chi(\varepsilon\mathbf{k}))}\,. \tag{2.2.31}$$

We can now prove that $N(\omega,\varepsilon\mathbf{k})$ is an analytic function of ω provided that $Im(\omega)\neq 0$. Firstly, any poles in the numerator $G_{J^tJ^t}(\omega,\varepsilon\mathbf{k})$, which can only occur in the lower half plane, will cancel out with those in the denominator. We thus need to check whether or not the denominator in (2.2.31) can vanish for $Im(\omega)\neq 0$. That this

[4]Note that for a superfluid one can have $\chi(\mathbf{k})$ diverging at $\mathbf{k}\to 0$.

cannot occur can be seen by writing

$$\frac{1}{i\omega}(G_{J^tJ^t}(\omega,\varepsilon\mathbf{k})+\chi(\varepsilon\mathbf{k}))=\int_{C_1}\frac{d\omega'}{i\pi}\frac{ImG_{J^tJ^t}(\omega',\varepsilon\mathbf{k})}{\omega'(\omega'-\omega)}\,, \tag{2.2.32}$$

where C_1 is a contour that skirts just under the real axis. Then writing $\omega = x + iy$, with $y \neq 0$, we can show that the real part of the integral is non-vanishing after using the fact that $ImG_{J^tJ^t}(\omega',\varepsilon\mathbf{k})/\omega' \leq 0$, which we showed in (2.2.16). We now return to (2.2.29), from which we deduce that, for fixed ω, as $\varepsilon \to 0$, we can expand

$$N=\varepsilon^2\frac{\mathbf{k}_i\mathbf{k}_j\,\sigma^{ij}(\omega)}{\chi(\mathbf{0})}+\dots\,, \tag{2.2.33}$$

with the neglected terms going to zero with a higher power of ε.

We are now in a position to discuss the poles of $G_{J^tJ^t}(\omega,\varepsilon\mathbf{k})$ that appear at the 'origin', by which we mean when both $\omega \to 0$ and $\varepsilon \to 0$. The simplest possibility is if $N(\omega,\varepsilon\mathbf{k})$ does not have any poles (or branch cuts) at $\omega = 0$. In this case, we see that when the DC conductivity matrix is finite, $G_{J^tJ^t}$ will have a single diffusion pole with dispersion relation

$$\omega=-i\varepsilon^2D(k)+\dots\,,\qquad D(k)=[\sigma^{ij}_{DC}k_ik_j]\chi(\mathbf{0})^{-1}\,, \tag{2.2.34}$$

and the neglected terms are higher order in ε. This is our first result on Einstein relations for inhomogeneous media.

It is important to emphasise that is not the only possibility. Indeed, as we discuss in the next subsection, there are additional poles when there are additional conserved currents. If, for example, we suppose that there are two conserved currents in total then a second diffusion pole can appear in $G_{J^tJ^t}(\omega,\varepsilon\mathbf{k})$. To illustrate this situation schematically, consider the behaviour of the following function for $\omega,\varepsilon\mathbf{k} \to 0$,

$$\varepsilon^2\left(\frac{A}{-i\omega+\varepsilon^2a}+\frac{B}{-i\omega+\varepsilon^2b}\right)\sim\frac{\varepsilon^2(A+B)}{-i\omega+\varepsilon^2\left(\frac{aA+bB}{A+B}-i\frac{AB(a-b)^2}{(A+B)^2}\frac{\varepsilon^2}{\omega}+\mathcal{O}(\frac{\varepsilon^2}{\omega})^2\right)}\,. \tag{2.2.35}$$

corresponding to the function $N(\omega,\varepsilon\mathbf{k})$ having additional singularities at $\omega \to 0$. Another interesting situation in which additional poles will appear is in the presence of Goldstone modes arising from broken symmetries. Additional general statements can be made using the memory matrix formalism, generalising the discussion in [131].

Returning now to the case in which there is just a single conserved current with a single diffusion pole then a natural phenomenological expression for the Green's function is given near the origin, $(\omega, \varepsilon\mathbf{k}) \to 0$, by

$$G_{J^tJ^t}(\omega, \varepsilon\mathbf{k}) \sim \frac{-D(\omega, \varepsilon\mathbf{k})}{-i\omega + D(\omega, \varepsilon\mathbf{k})}\chi(\varepsilon\mathbf{k})\,, \tag{2.2.36}$$

with $D(\omega, \varepsilon\mathbf{k}) \sim \varepsilon^2 \frac{\mathbf{k}_i\mathbf{k}_j\,\sigma^{ij}(\omega)}{\chi(\varepsilon\mathbf{k})}$. It is interesting to note that if, by contrast, we are in the context of infinite DC conductivity with $\sigma^{ij}(\omega) \sim K^{ij}\,\left(\frac{i}{\omega} + \pi\,\delta(\omega)\right)$ for small ω, where K^{ij} is constant, then (2.2.36) gives rise to sound modes for the current density J^t, with dispersion relation $\omega_\pm = \pm\varepsilon\sqrt{K^{ij}\,\mathbf{k}_i\mathbf{k}_j/\chi(\varepsilon\mathbf{k})}$. The transition between diffusion modes and sound modes was also discussed in a homogeneous hydrodynamic setting, with a phenomenological term to relax momentum, in [238].

To conclude this subsection, we briefly note that we can carry out a similar analysis for the higher Fourier modes of the current-current correlators. Starting with (2.2.21), the analogue of (2.2.27) is

$$\frac{1}{i\omega}(\mathbf{k} + n_r\mathbf{k}_L^r)_i\,\mathbf{k}_j\,G^{(\{n_l\})}_{J^i_AJ^j_B}(\omega, \mathbf{k}) = -i\omega\,G^{(\{n_l\})}_{J^t_AJ^t_B}(\omega, \mathbf{k})\,, \tag{2.2.37}$$

and this leads, *mutatis-mutandis*, to additional relations concerning the poles of $G^{(\{n_l\})}_{J^t_AJ^t_B}(\omega, \mathbf{k})$, which would be interesting to explore in more detail. We note however, that for $\{n_l\} \neq 0$, there is no longer a simple statement concerning the positivity of $Im G^{(\{n_l\})}_{J^t_AJ^t_B}(\omega, \mathbf{k})/\omega$, which was used in the above. We also point out that within a holographic context and for a specific gravitational model, some of the $G^{(\{n_l\})}_{J^i_AJ^j_B}(\omega, \mathbf{k})$ were calculated in [285].

2.2.2 Generalised Einstein relations for multiple currents

We now assume that we have multiple conserved currents J^μ_A. For example, one could have both a conserved heat current and a conserved $U(1)$ current. Much of the analysis that we carried out for the case of a single current goes through straightforwardly and we obtain

$$\frac{1}{i\omega}\mathbf{k}_i\mathbf{k}_j\,G_{J^i_AJ^j_B}(\omega, \varepsilon\,\mathbf{k}) = -\frac{i\omega}{\varepsilon^2}\,G_{J^t_AJ^t_B}(\omega, \varepsilon\mathbf{k})\,. \tag{2.2.38}$$

We write the charge susceptibilities and the AC conductivity via

$$\begin{aligned}\boldsymbol{\chi}_{AB}(\varepsilon\mathbf{k}) &= -\lim_{\omega\to 0+i0} G_{J^t_AJ^t_B}(\omega, \varepsilon\mathbf{k})\,,\\ \sigma^{ij}_{AB}(\omega) &= -\lim_{\varepsilon\to 0}\frac{1}{i\omega}G_{J^i_AJ^j_B}(\omega, \varepsilon\,\mathbf{k})\,,\end{aligned} \tag{2.2.39}$$

respectively, and we now have

$$\mathbf{k}_i\mathbf{k}_j\,\sigma^{ij}_{AB}(\omega) = i\omega\,\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\,G_{J^t_AJ^t_B}(\omega, \varepsilon\mathbf{k})\,. \tag{2.2.40}$$

Generically this shows that for finite DC conductivities there will be at least as many poles in the transport current correlators as there are currents.

Proceeding much as before we write

$$\mathbf{G}(\omega, \varepsilon\mathbf{k})\boldsymbol{\chi}(\varepsilon\mathbf{k})^{-1} = -\left[-i\omega + \mathbf{N}(\omega, \varepsilon\mathbf{k})\right]^{-1} \mathbf{N}(\omega, \varepsilon\mathbf{k}) \,, \tag{2.2.41}$$

where $\mathbf{G}(\omega, \varepsilon\mathbf{k})_{AB} \equiv G_{J^t_A J^t_B}(\omega, \varepsilon\mathbf{k})$ and

$$\mathbf{N}(\omega, \varepsilon\mathbf{k}) \equiv \mathbf{G}(\omega, \varepsilon\mathbf{k}) \left[\frac{1}{i\omega}(\mathbf{G}(\omega, \varepsilon\mathbf{k}) + \boldsymbol{\chi}(\varepsilon\mathbf{k}))\right]^{-1} . \tag{2.2.42}$$

We can again argue that $\mathbf{N}(\omega, \varepsilon\mathbf{k})$ can only have poles on the real ω axis. From (2.2.40) we deduce that for fixed ω, as $\varepsilon \to 0$, we can expand

$$\mathbf{N}(\omega, \varepsilon\mathbf{k}) = \varepsilon^2 \boldsymbol{\Sigma}(\omega, \mathbf{k})\boldsymbol{\chi}(\varepsilon\mathbf{k})^{-1} \,, \tag{2.2.43}$$

where $\boldsymbol{\Sigma}(\omega, \mathbf{k})_{AB} = \mathbf{k}_i \mathbf{k}_j \, \sigma^{ij}_{AB}(\omega)$ and the neglected terms go to zero with a higher power of ε.

If we now assume that $\mathbf{N}(\omega, \varepsilon\mathbf{k})$ doesn't have any poles at $\omega = 0$, then we can conclude that at the origin, i.e. when both $\omega \to 0$ and $\varepsilon \to 0$, if the DC conductivities are finite then the diffusion poles of the system are located at

$$\omega_A(\mathbf{k}) = -i \, D_A(\mathbf{k}) \, \varepsilon^2 + \cdots \,, \tag{2.2.44}$$

where $D_A(\mathbf{k})$ are the eigenvalues of what can be called the 'generalised diffusion matrix' $\mathbf{D}(\mathbf{k})$ defined by

$$\mathbf{D}(\mathbf{k}) = \boldsymbol{\Sigma}(0, \mathbf{k})\boldsymbol{\chi}(\mathbf{0})^{-1} \,, \tag{2.2.45}$$

and the dots involve higher order corrections in ε. In particular when the DC conductivities are finite, the number of diffusion poles is the same as the number of conserved currents.

Furthermore, we emphasise that when there is more than one conserved current, generically, these diffusion modes do not satisfy a dispersion relation of the form $\omega \sim -i\varepsilon^2 \Sigma_{ij} k^i k^j$, with the matrix Σ_{ij} a component of the DC conductivities. As a consequence we refer to our result (2.2.44),(2.2.45) as a 'generalised Einstein relation'.

We conclude this section by noting that the general result (2.2.44), (2.2.45) relates thermodynamic instabilities to dynamic instabilities. Suppose that the system has a static susceptibility matrix $\boldsymbol{\chi}(\mathbf{0})$ with a negative eigenvalue and hence is thermodynamically unstable. Then (2.2.45) implies that $\mathbf{D}(\mathbf{k})$ will have a negative eigenvalue, for small $\mathbf{k}$, and hence, from (2.2.44) we deduce that there will be a diffusion pole in the upper half plane leading to a dynamical instability[5].

2.3 Diffusion in relativistic hydrodynamics

We now discuss thermoelectric transport within the context of relativistic hydrodynamics. As well as generalising the work of [300] to include a conserved $U(1)$ charge (as also studied in [304]), we will also be able to use the formalism to illustrate the results of the previous section. In particular, associated with the heat current and the $U(1)$ current we construct two diffusion modes with dispersion relations satisfying the generalised Einstein relation (2.2.44). We note that it will be convenient to use a slightly different notation in this section, which implies that a little care is required in directly comparing with the last section.

2.3.1 General setup

We will consider an arbitrary relativistic quantum field theory with a global $U(1)$ symmetry in $d \geq 2$ spacetime dimensions. The field theory is defined on a static, curved manifold, with metric $g_{\mu\nu}$, and a non-zero background gauge-field, A_μ, of the form:

$$\begin{aligned} ds^2 &= -f^2(x)dt^2 + h_{ij}(x)dx^i dx^j \,, \\ A_t &= a_t(x) \,. \end{aligned} \tag{2.3.1}$$

This corresponds to studying the field theory with f^2 and h_{ij} parametrising sources for the stress tensor components T^{tt} and T^{ij}, respectively, and a_t parametrising a source for the J^t component of the conserved $U(1)$ current. We focus on cases in which the manifold has planar topology, with the globally defined spatial coordinates x^i parametrising $\mathbb{R}^{d-1}$, and f, h_{ij},a_t all depending periodically on x^i, with period L_i.

We will study the field theory at finite temperature in the hydrodynamic limit keeping the leading order viscous terms. In particular, we will consider temperatures[6] that are much greater than the largest wave-number that appears in the background fields in (2.3.1). The Ward identities are given by

$$D_\mu T^{\mu\nu} = F^{\nu\lambda} J_\lambda \,, \qquad D_\mu J^\mu = 0 \,, \tag{2.3.2}$$

where D_μ is the covariant derivative with respect to $g_{\mu\nu}$ and $F_{\mu\nu} = 2\partial_{[\mu} A_{\nu]}$. For

[5] An explicit example of such a dynamic instability can be seen using the results of appendix 2.B.

[6] This temperature is the same as what is denoted as $\bar{T}_0$ below.

the special case of conformal field theory, we should also impose $T^\mu{}_\mu = 0$ and this implies, amongst other things, that in (2.3.3) $\zeta_b = 0$ and $\epsilon = (d-1)P$.

The hydrodynamic variables are the local temperature, $T(x)$, the local chemical potential, $\mu(x)$, and the fluid velocity, u^μ, with $u^\mu u^\nu g_{\mu\nu} = -1$. As in [165], the constitutive relations are given, in the Landau frame, by[7]

$$\begin{aligned}
T_{\mu\nu} =& P g_{\mu\nu} + (P+\epsilon) u_\mu u_\nu - 2\eta \left(D_{(\mu} u_{\nu)} + u_\rho u_{(\mu} D^\rho u_{\nu)} - (g_{\mu\nu} + u_\mu u_\nu) \frac{D_\rho u^\rho}{d-1} \right) \\
&- \zeta_b (g_{\mu\nu} + u_\mu u_\nu) D_\rho u^\rho \,, \\
J^\mu =& \rho u^\mu + \sigma_Q \left(F^{\mu\nu} u_\nu - T(g^{\mu\nu} + u^\mu u^\nu) D_\nu \left(\frac{\mu}{T} \right) \right) \,,
\end{aligned} \tag{2.3.3}$$

where P is the pressure density, ϵ is the energy density and ρ is the $U(1)$ charge density. The dissipative terms in (2.3.3) are the shear viscosity, η, the bulk viscosity, ζ_b and the conductivity, σ_Q, which should not be confused with the electrical DC conductivity, σ_{DC}, which we discuss later. We also have the local thermodynamic relation and first law which take the form

$$P + \epsilon = sT + \mu\rho, \qquad dP = s dT + \rho d\mu \,, \tag{2.3.4}$$

where s is the entropy density. It will also be helpful to introduce the susceptibilities c_μ, ξ and χ via

$$ds = T^{-1} c_\mu \, dT + \xi \, d\mu \,, \qquad d\rho = \xi \, dT + \chi \, d\mu \,. \tag{2.3.5}$$

For any vector k, the Ward identities imply

$$D_\mu [(T^\mu{}_\nu + J^\mu A_\nu) k^\nu] = \frac{1}{2} \mathcal{L}_k g_{\mu\nu} T^{\mu\nu} + \mathcal{L}_k A_\mu J^\mu \,, \tag{2.3.6}$$

where $\mathcal{L}_k$ is the Lie derivative. Taking $k = \partial_t$ we define the heat current as

$$Q^\mu = -(T^\mu{}_t + A_t J^\mu) \,, \tag{2.3.7}$$

which is conserved for stationary metrics with $\mathcal{L}_k A_\nu = 0$. Thus, given such background metrics and gauge fields, for time independent configurations we therefore have $\partial_i(\sqrt{-g}\, Q^i) = \partial_i(\sqrt{-g}\, J^i) = 0$.

In thermal equilibrium the fluid configuration is given by

$$u_t = -f(x) \,, \qquad u_i = 0 \,, \qquad T = T_0(x) \,, \qquad \mu = \mu_0(x) \,, \tag{2.3.8}$$

where $T_0(x)$ and $\mu_0(x)$ are periodic functions, and from (2.3.4) we have the equilib-

[7] Following [165], we have set to zero two other terms in J^μ that are allowed by Lorentz invariance but are not consistent with positivity of entropy and thermodynamics with external sources.

rium relations

$$P_0 + \epsilon_0 = s_0 T_0 + \mu_0 \rho_0, \qquad \partial_i P_0 = s_0 \partial_i T_0 + \rho_0 \partial_i \mu_0 \,. \tag{2.3.9}$$

For later use, we note that we also have

$$\nabla_i s_0 = T_0^{-1} c_{\mu 0}\, \nabla_i T_0 + \xi_0\, \nabla_i \mu_0 \,, \qquad \nabla_i \rho_0 = \xi_0 \nabla_i T_0 + \chi_0\, \nabla_i \mu_0 \,. \tag{2.3.10}$$

By calculating $T^{\mu\nu}$, J^μ one can show that the Ward identities are satisfied provided that

$$T_0 = f^{-1} \bar{T}_0 \,, \qquad \mu_0 = f^{-1} a_t \,, \tag{2.3.11}$$

where $\bar{T}_0$ is constant. Note, in particular, that in thermal equilibrium the local hydrodynamic variable T_0 is not constant when f is not constant and, furthermore, there is a factor of f that appears in the relationship between μ_0 and the background gauge field. We also note that we have set a possible integration constant to zero in the second expression as we want μ_0 to vanish when a_t does. Finally it will be helpful to define the zero mode of a_t via $\bar{\mu}_0 \equiv \oint a_t$, where we are again using the notation $\oint \equiv (L_1 \cdots L_d)^{-1} \int_{\{0\}}^{\{\mathbf{L_i}\}} dx^1 \cdots dx^d$. This allows us to write $\mu_0 = f^{-1}(\bar{\mu}_0 + \tilde{a}_t(x))$, with $\oint \tilde{a}_t = 0$.

The non-vanishing components of the stress tensor and current for this equilibrium configuration are then given by

$$T_{tt} = \epsilon_0 f^2, \qquad T_{ij} = P_0 h_{ij}, \qquad J^t = \rho_0 f^{-1} \,. \tag{2.3.12}$$

In particular for the backgrounds we are considering, in thermal equilibrium both the electric and the heat currents vanish: $J^i = Q^i = 0$. Note, since (2.3.1) provides a source for the energy and the charge, we can immediately deduce that the charge-current susceptibilities must vanish. The total energy and charge of the equilibrium configuration are defined by

$$\begin{aligned} \epsilon_{tot} &= -\oint \sqrt{-g} T^t{}_t = \oint \sqrt{h} f \epsilon_0 \,, \\ \rho_{tot} &= \oint \sqrt{-g} J^t = \oint \sqrt{h} \rho_0 \,. \end{aligned} \tag{2.3.13}$$

We can also define the total equilibrium entropy as

$$s_{tot} = \oint \sqrt{h} s_0 \,. \tag{2.3.14}$$

For later use, using the fact that s_0 is a function of T_0 and μ_0, we observe that for suitable zero modes of the charge susceptibilities we have

$$\frac{\partial s_{tot}}{\partial \bar{T}_0} = \oint \sqrt{h} f^{-1} T_0^{-1} c_{\mu 0}\,, \qquad \frac{\partial s_{tot}}{\partial \bar{\mu}_0} = \oint \sqrt{h} f^{-1} \xi_0\,. \tag{2.3.15}$$

Similarly, we also have

$$\frac{\partial \rho_{tot}}{\partial \bar{T}_0} = \oint \sqrt{h} f^{-1} \xi_0\,, \qquad \frac{\partial \rho_{tot}}{\partial \bar{\mu}_0} = \oint \sqrt{h} f^{-1} \chi_0\,. \tag{2.3.16}$$

2.3.2 Generalised Navier-Stokes equations

In the following we want to study the behaviour of small perturbations about the equilibrium configuration, including the possibility of adding external, perturbative thermal gradient and electric field sources. Following [300] we will do this by considering

$$\begin{aligned} ds^2 &= -f^2(1-2\phi_T)\,dt^2 + h_{ij}dx^i dx^j\,, \\ A_t &= a_t - f\mu_0\phi_T + \phi_E\,, \end{aligned} \tag{2.3.17}$$

along with

$$\begin{aligned} u_t &= -f(1-\phi_T)\,, \qquad & u_i &= \delta u_i\,, \\ T &= T_0 + \delta T\,, \qquad & \mu &= \mu_0 + \delta\mu\,. \end{aligned} \tag{2.3.18}$$

Here ϕ_T, ϕ_E, δu_i, δT and $\delta\mu$ are all functions of (t, x^i). Note that these need not be periodic functions of the spatial coordinates. For later use, we also define the spatial components of the external sources ζ_i, E_i via

$$\zeta_i = \partial_i \phi_T, \qquad E_i = \partial_i \phi_E\,. \tag{2.3.19}$$

At linearised order, the perturbed stress tensor and $U(1)$ current can then be written as

$$\begin{aligned} T_{tt} &= \epsilon_0 f^2(1-2\phi_T) + \delta\epsilon\, f^2\,, \\ T_{ti} &= -f(P_0+\epsilon_0)\delta u_i\,, \\ T_{ij} &= (P_0+\delta P)h_{ij} - 2\eta_0 f^{-1}\left(\nabla_{(i}\left(f\delta u_{j)}\right) - \frac{h_{ij}}{(d-1)}\nabla_k\left(f\delta u^k\right)\right) \\ &\quad - \zeta_{b0} h_{ij} f^{-1}\nabla_k\left(f\delta u^k\right)\,, \\ J^t &= \rho_0 f^{-1}(1+\phi_T) + f^{-1}\delta\rho\,, \\ J^i &= \rho_0\delta u^i + \sigma_{Q0} f^{-1}\left[E^i - \nabla^i\left(f\delta\mu\right) - f\mu_0\zeta^i + \mu_0 T_0^{-1}\nabla^i\left(f\delta T\right)\right]\,, \end{aligned} \tag{2.3.20}$$

where ∇_i is the covariant derivative with respect to the metric h_{ij}, which is also used to raise and lower indices. The Ward identities (2.3.2) give

$$\begin{aligned}
&\partial_t\delta\rho+\nabla_i\left(fJ^i\right)=0\,,\\
&f\partial_t\delta\epsilon+\nabla_i\left(f^2(P_0+\epsilon_0)\delta u^i\right)-fJ^i\nabla_i a_t=0\,,\\
&f^{-1}(P_0+\epsilon_0)\partial_t\delta u_j-2f^{-1}\nabla^i\left(\eta_0\nabla_{(i}\left(f\delta u_{j)}\right)\right)\\
&+f^{-1}\nabla_j\left(\left(\frac{2\eta_0}{(d-1)}-\zeta_{b0}\right)\nabla_k\left(f\delta u^k\right)\right)\\
&=-\nabla_j\delta P-(\delta\epsilon+\delta P)f^{-1}\nabla_j f+(P_0+\epsilon_0)\zeta_j+\rho_0(f^{-1}E_j-\mu_0\zeta_j)+f^{-1}\delta\rho\nabla_j a_t\,,
\end{aligned}\tag{2.3.21}$$

In the case when there is no $U(1)$ charge this agrees with the expression derived in equation (A.10) of [300]. These expressions can be further simplified. We use (2.3.11) as well as

$$\begin{aligned}
\delta P&=s_0\delta T+\rho_0\delta\mu\,, &\qquad \delta\epsilon&=T_0\delta s+\mu_0\delta\rho\,,\\
\delta s&=T_0^{-1}c_{\mu0}\,\delta T+\xi_0\,\delta\mu, &\qquad \delta\rho&=\xi_0\,\delta T+\chi_0\,\delta\mu\,,
\end{aligned}\tag{2.3.22}$$

which we obtain from (2.3.4),(2.3.5). After also using (2.3.10) we eventually find that we can rewrite the system (2.3.21) in the following form, which is the key result of this section,

$$\begin{aligned}
\xi_0\partial_t\delta T+\chi_0\partial_t\delta\mu+\nabla_i\left(fJ^i\right)&=0\,,\\
fc_{\mu0}\partial_t\delta T+fT_0\xi_0\partial_t\delta\mu+\nabla_i\left(fQ^i\right)&=0\,,\\
(P_0+\epsilon_0)\partial_t\delta u_j-2\nabla^i\left(\eta_0\nabla_{(i}\left(f\delta u_{j)}\right)\right)+\nabla_j\left(\left(\frac{2\eta_0}{(d-1)}-\zeta_{b0}\right)\nabla_k\left(f\delta u^k\right)\right)&=\\
\rho_0\left[E_j-\nabla_j\left(f\delta\mu\right)\right]+fT_0s_0\left[\zeta_j-(fT_0)^{-1}\nabla_j\left(f\delta T\right)\right]&,
\end{aligned}\tag{2.3.23}$$

with

$$\begin{aligned}
J^i&=\rho_0\delta u^i+\sigma_{Q0}f^{-1}\left[E^i-\nabla^i\left(f\delta\mu\right)\right]-\sigma_{Q0}\mu_0\left[\zeta^i-(fT_0)^{-1}\nabla^i\left(f\delta T\right)\right]\,,\\
Q^i&=f(P_0+\epsilon_0)\delta u^i-f\mu_0J^i\,.
\end{aligned}\tag{2.3.24}$$

Notice that the first two lines in (2.3.23) are just current conservation equations for the linearised perturbation. We emphasise that all background equilibrium quantities, marked with a 0 subscript, are all periodic functions of the spatial coordinates. It is interesting to note that the system of equations (2.3.23) is invariant under the interchange

$$E_j\leftrightarrow-\nabla_j\left(f\delta\mu\right)\,,\qquad \zeta_j\leftrightarrow-f^{-1}T_0^{-1}\nabla_j\left(f\delta T\right)\,.\tag{2.3.25}$$

Finally, for later use, we note that when the sources are set to zero, $\phi_T = \phi_E = 0$, we have for the total charges

$$\begin{aligned} \oint \sqrt{-g} J^t &= \oint \sqrt{h}\rho_0 + \oint \sqrt{h}\delta\rho\,, \\ \oint \sqrt{-g} Q^t &= \oint \sqrt{h} f(\epsilon_0 - \mu_0\rho_0) + \bar{T}_0 \oint \sqrt{h}\delta s\,. \end{aligned} \tag{2.3.26}$$

2.3.3 Thermoelectric DC conductivity

We now explain how we can obtain the thermoelectric DC conductivity, generalising [300]. We begin by considering the sources ϕ_T and ϕ_E to have space and time dependence of the form $e^{-i\omega t}e^{ik_i x^i}$, where k_i is an arbitrary wave number. After solving (2.3.23) for δu_j, $\delta\mu$, δT one obtains the local currents J^i Q^i, and hence the current fluxes $\bar{J}^i$ $\bar{Q}^i$, as functions of E_i and ζ_i. To obtain the thermoelectric DC conductivity we should then take the limit $k_i \to 0$, followed by $\omega \to 0$.

By considering approximating $e^{ik_i x^i} \sim 1 + ik_i x^i$ we are prompted[8] to consider a time-independent source of the form

$$\phi_T = x^i \bar{\zeta}_i, \qquad \phi_E = x^i \bar{E}_i\,, \tag{2.3.27}$$

where $\bar{\zeta}_i$, $\bar{E}_i$ are constants and hence $E_i = \bar{E}_i$, $\zeta_i = \bar{\zeta}_i$. After substituting into (2.3.32) we obtain the system[9]

$$\begin{aligned} &\nabla_i \left(f J^i\right) = 0\,, \qquad \nabla_i \left(f Q^i\right) = 0\,, \\ &-2\nabla^i \left(\eta_0 \nabla_{(i} \left(f \delta u_{j)}\right)\right) + \nabla_j \left(\left(\frac{2\eta_0}{(d-1)} - \zeta_{b0}\right) \nabla_k \left(f\delta u^k\right)\right) = \\ &\qquad \rho_0 \bar{E}_j - \rho_0 \nabla_j \left(f\delta\mu\right) + f s_0 T_0 \bar{\zeta}_j - s_0 \nabla_j \left(f\delta T\right) \qquad . \end{aligned} \tag{2.3.28}$$

After solving these equations we obtain the local time-independent, steady state currents $J^i(x)$ $Q^i(x)$, periodic in the spatial coordinates, as functions of $\bar{\zeta}_i$, $\bar{E}_i$. We can now define the heat and charge current fluxes via

$$\bar{Q}^i \equiv \oint \sqrt{-g} Q^i = \oint \sqrt{h} f Q^i\,, \qquad \bar{J}^i \equiv \oint \sqrt{-g} J^i = \oint \sqrt{h} f J^i\,, \tag{2.3.29}$$

[8] An alternative procedure is to consider sources that are linear in time, as explained in a holographic context in [222, 289].

[9] In the special case of conformal field theories, similar equations were obtained in a holographic context in [290]. The equations differ when there is a $U(1)$ symmetry due to a difference in the expression for Q^i in (2.3.24). The equations should agree in the hydrodynamic limit, after a possible change of frame and/or incorporating higher order terms in the hydrodynamic expansion, and it would be interesting to investigate this in more detail.

and the DC conductivities are obtained from

$$\begin{pmatrix} \bar{J}^i \\ \bar{Q}^i \end{pmatrix} = \begin{pmatrix} \sigma^{ij}_{DC} & \bar{T}_0 \alpha^{ij}_{DC} \\ \bar{T}_0 \bar{\alpha}^{ij}_{DC} & \bar{T}_0 \bar{\kappa}^{ij}_{DC} \end{pmatrix} \begin{pmatrix} \bar{E}_j \\ \bar{\zeta}_j \end{pmatrix} . \tag{2.3.30}$$

Since we are considering backgrounds which preserve time reversal invariance the Onsager relations imply that σ_{DC} and $\bar{\kappa}_{DC}$ are symmetric matrices and $\alpha^T_{DC} = \bar{\alpha}_{DC}$.

2.3.4 Diffusive modes

We now discuss how we can construct a perturbative diffusive solution of the system of equations (2.3.23) that is associated with diffusion modes. Our objective will be to extract the associated dispersion relations for these modes.

We first set the source terms in (2.3.23) to zero: $E_i = \zeta_i = 0$. We will allow for a time-dependence of the form $e^{-i\omega t}$ and consider the expansion

$$\omega = \sum_{\alpha=1}^{\infty} \varepsilon^{\alpha} \omega^{(\alpha)} , \tag{2.3.31}$$

with $\varepsilon \ll 1$. Since we are interested in wavelengths that are much larger than the periods, L_i, of the background fields in (2.3.1), we introduce arbitrary wave numbers k^i and consider

$$\delta T = e^{-i\omega t} e^{i\varepsilon k_i x^i} \sum_{\alpha=0}^{\infty} \varepsilon^{\alpha} \delta T^{(\alpha)}(x) , \qquad \delta\mu = e^{-i\omega t} e^{i\varepsilon k_i x^i} \sum_{\alpha=0}^{\infty} \varepsilon^{\alpha} \delta\mu^{(\alpha)}(x) ,$$
$$\delta u_i = e^{-i\omega t} e^{i\varepsilon k_i x^i} \sum_{\alpha=0}^{\infty} \varepsilon^{\alpha} \delta u_i^{(\alpha)}(x) , \tag{2.3.32}$$

with the functions inside the summations taken to be periodic in the x^i, with period L^i.

We next note that the system of equations (2.3.23) (with $E^i = \zeta^i = 0$) admit the simple time-independent solution with $f\,\delta T$, $f\,\delta\mu$ both constant and $\delta u_i = 0$. Indeed, from (2.3.11) this corresponds to simply perturbing the parameters of the thermal equilibrium configuration. The diffusive modes are constructed as a perturbation of this time-independent solution by using the expansions (2.3.31), (2.3.32) and taking

$$f\,\delta T^{(0)} = constant, \qquad f\,\delta\mu^{(0)} = constant, \qquad \delta u_i^{(0)} = 0 , \tag{2.3.33}$$

as the zeroth order solution. We immediately see that the associated expansion for J^i and Q^i can be written as

$$J^i = e^{-i\omega t} e^{i\varepsilon k_i x^i} \sum_{\alpha=1}^{\infty} \varepsilon^{\alpha} J^{i(\alpha)}(x) , \qquad Q^i = e^{-i\omega t} e^{i\varepsilon k_i x^i} \sum_{\alpha=1}^{\infty} \varepsilon^{\alpha} Q^{i(\alpha)}(x) . \tag{2.3.34}$$

At leading order in ε, the first two equations of (2.3.23) then read

$$-i\omega^{(1)}\xi_0\delta T^{(0)} - i\omega^{(1)}\chi_0\delta\mu^{(0)} + \nabla_i\left(fJ^{i(1)}\right) = 0\,,$$
$$-i\omega^{(1)}c_{\mu 0}f\delta T^{(0)} - i\omega^{(1)}T_0\xi_0 f\delta\mu^{(0)} + \nabla_i\left(fQ^{i(1)}\right) = 0\,. \qquad (2.3.35)$$

Integrating equations (2.3.35) over a period we obtain

$$i\omega^{(1)}\oint\sqrt{h}\left(\xi_0\,\delta T^{(0)} + \chi_0\,\delta\mu^{(0)}\right) = 0\,,$$
$$i\omega^{(1)}\oint\sqrt{h}f\left(c_{\mu 0}\,\delta T^{(0)} + T_0\xi_0\,\delta\mu^{(0)}\right) = 0\,. \qquad (2.3.36)$$

Assuming thermodynamically stable matter, the matrix of static susceptibilities, whose components appear in (2.3.36), is positive definite and these equations can only be satisfied by setting $\omega^{(1)} = 0$. The leading order system (2.3.23) then becomes

$$\nabla_i\left(fJ^{i(1)}\right) = 0\,, \qquad \nabla_i\left(fQ^{i(1)}\right) = 0\,,$$
$$-2\nabla^i\left(\eta_0\nabla_{(i}\left(f\delta u^{(1)}_{j)}\right)\right) + \nabla_j\left(\left(\frac{2\eta_0}{(d-1)} - \zeta_{b0}\right)\nabla_k\left(f\delta u^{k(1)}\right)\right) =$$
$$-i\rho_0 k_j f\delta\mu^{(0)} - \rho_0\nabla_j\left(f\delta\mu^{(1)}\right) - is_0 k_j f\delta T^{(0)} - s_0\nabla_j\left(f\delta T^{(1)}\right)\,. \qquad (2.3.37)$$

with

$$J^{i(1)} = \rho_0\delta u^{i(1)} + \sigma_{Q0}f^{-1}\left[-\nabla^i\left(f\delta\mu^{(1)}\right)\right] - \sigma_{Q0}\mu_0\left[-(fT_0)^{-1}\nabla^i\left(f\delta T^{(1)}\right)\right]\,,$$
$$Q^{i(1)} = f(P_0+\epsilon_0)\delta u^{i(1)} - f\mu_0 J^{i(1)}\,. \qquad (2.3.38)$$

Notice that this system is equivalent to the system of equations (2.3.28) that appeared for the calculation of the thermoelectric DC conductivity if we identify $\bar{E}_i \leftrightarrow -ik_i f\delta\mu^{(0)}$, $\bar{\zeta}_i \leftrightarrow -ik_j T_0^{-1}\delta T^{(0)}$ and note that the quantities on the right hand sides of these expressions are indeed constant. Thus, we can express the heat current fluxes $\bar{J}^{i(1)}$ and $\bar{Q}^{i(1)}$ in terms of $-ik_i f\delta\mu^{(0)}$, $-ik_j T_0^{-1}\delta T^{(0)}$ using the thermoelectric DC conductivity matrix given in (2.3.30) to get

$$\bar{J}^{i(1)} \equiv \oint\sqrt{h}fJ^{i(1)} = -i\sigma^{ij}_{DC}k_j\,f\delta\mu^{(0)} - i\alpha^{ij}_{DC}k_j\,f\delta T^{(0)}\,,$$
$$\bar{Q}^{i(1)} \equiv \oint\sqrt{h}fQ^{i(1)} = -i\bar{T}_0\alpha^{ij}_{DC}k_j\,f\delta\mu^{(0)} - i\bar{\kappa}^{ij}_{DC}k_j\,f\delta T^{(0)}\,. \qquad (2.3.39)$$

Continuing the expansion, we next examine the first two equations of (2.3.23) at second order in ε to find

$$-i\omega^{(2)}\xi_0\delta T^{(0)} - i\omega^{(2)}\chi_0\delta\mu^{(0)} + ik_i fJ^{i(1)} + \nabla_i\left(fJ^{i(2)}\right) = 0\,,$$
$$-i\omega^{(2)}c_{\mu 0}f\delta T^{(0)} - i\omega^{(2)}T_0\xi_0 f\delta\mu^{(0)} + ik_i fQ^{i(1)} + \nabla_i\left(fQ^{i(2)}\right) = 0\,. \qquad (2.3.40)$$

Integrating these two equations over a period, substituting the expression for the

DC conductivity and using (2.3.15),(2.3.16) we now deduce

$$\begin{aligned} i\omega^{(2)}\left(\frac{\partial\rho_{tot}}{\partial\bar{T}_0}f\delta T^{(0)}+\frac{\partial\rho_{tot}}{\partial\bar{\mu}_0}f\delta\mu^{(0)}\right)-\alpha^{ij}_{DC}k_ik_j\,f\delta T^{(0)}-\sigma^{ij}_{DC}k_ik_j\,f\delta\mu^{(0)}&=0\,,\\ i\omega^{(2)}\bar{T}_0\left(\frac{\partial s_{tot}}{\partial\bar{T}_0}f\delta T^{(0)}+\frac{\partial s_{tot}}{\partial\bar{\mu}_0}f\delta\mu^{(0)}\right)-\bar{\kappa}^{ij}_{DC}k_ik_j\,f\delta T^{(0)}-\bar{T}_0\alpha^{ij}_{DC}k_ik_j\,f\delta\mu^{(0)}&=0\,. \end{aligned} \tag{2.3.41}$$

Writing this in matrix form as

$$\mathbb{M}\begin{pmatrix} f\delta T^{(0)}\\ f\delta\mu^{(0)}\end{pmatrix}=0\,, \tag{2.3.42}$$

we have $\det(\mathbb{M})=0$. This gives rises to a quadratic equation for $i\omega^{(2)}$ which has two solutions, $i\omega^{(2)}_{\pm}$, which give the leading order dispersion relations for the diffusion modes that we are after.

To write $i\omega^{(2)}_{\pm}$ in a compact way we first define the scalar quantities depending on the DC conductivities that are quadratic in the wave numbers k^i:

$$\bar{\kappa}(k)\equiv\bar{\kappa}^{ij}_{DC}k_ik_j\,,\qquad \alpha(k)\equiv\alpha^{ij}_{DC}k_ik_j\,,\quad \sigma(k)\equiv\sigma^{ij}_{DC}k_ik_j\,, \tag{2.3.43}$$

as well as

$$\kappa(k)\equiv\bar{\kappa}(k)-\frac{\alpha(k)^2\bar{T}_0}{\sigma(k)}\,. \tag{2.3.44}$$

Recall that $\kappa^{ij}_{DC}\equiv\bar{\kappa}^{ij}_{DC}-\bar{T}_0(\bar{\alpha}_{DC}\cdot\sigma^{-1}_{DC}\cdot\alpha_{DC})^{ij}$ is the DC thermal conductivity for zero electric current and in general $\kappa(k)\neq\kappa^{ij}_{DC}k^ik^j$. We also define the following susceptibilities:

$$X=\frac{\partial\rho_{tot}}{\partial\bar{\mu}_0}\,,\qquad \Xi=\frac{\partial s_{tot}}{\partial\bar{\mu}_0}=\frac{\partial\rho_{tot}}{\partial\bar{T}_0},\qquad C_\rho=\oint\sqrt{h}c_{\mu0}-\frac{\bar{T}_0\Xi^2}{X}\,. \tag{2.3.45}$$

Note that if we consider the susceptibility $c_\rho=T(\partial s/\partial T)_\rho=c_\mu-\frac{T\xi^2}{\chi}$, in general $C_\rho\neq\oint\sqrt{h}c_{\rho0}$. Using these definitions, we then find that

$$\begin{aligned} i\omega^{(2)}_{+}\,i\omega^{(2)}_{-}&=\frac{\kappa(k)}{C_\rho}\frac{\sigma(k)}{X}\,,\\ i\omega^{(2)}_{+}+i\omega^{(2)}_{-}&=\frac{\kappa(k)}{C_\rho}+\frac{\sigma(k)}{X}+\frac{\bar{T}_0\,\left(X\,\alpha(k)-\Xi\,\sigma(k)\right)^2}{C_\rho X^2\sigma(k)}\,. \end{aligned} \tag{2.3.46}$$

This is the main result of this section and it should be compared with the general result given in (2.2.44),(2.2.45) that we obtained in the previous section.

A number of comments are in order. Firstly, for relativistic hydrodynamics without a $U(1)$ current, there is just a single energy diffusion mode. In this case, the leading order dispersion relation is given by

$$i\omega^{(2)} = \frac{\kappa^{ij}_{DC} k_i k_j}{\bar{T}_0 \frac{\partial s_{tot}}{\partial \bar{T}_0}}\,. \tag{2.3.47}$$

This result should be compared with (2.2.34). Similarly, we can also consider charge neutral backgrounds which have $\Xi = \alpha^{ij}_{DC} = 0$ and then the equations (2.3.41) decouple. In particular we find a charge diffusion mode with leading order dispersion relation given by

$$i\omega^{(2)} = \frac{\sigma^{ij}_{DC} k_i k_j}{\frac{\partial \rho_{tot}}{\partial \bar{\mu}_0}}\,. \tag{2.3.48}$$

Our next comment concerns perturbative lattices. By definition a perturbative lattice is one in which the metric and gauge field deformations have a perturbatively small amplitude. In this case the spatial momentum dissipation is weak. Using the memory matrix formalism [117] or holography [290] we have

$$\bar{\kappa}^{ij}_{DC} = 4\pi s_0 T_0 L^{-1}_{ij}, \quad \alpha^{ij}_{DC} = 4\pi \rho_0 L^{-1}_{ij}, \quad \sigma^{ij}_{DC} = 4\pi s_0^{-1} \rho_0^2 L^{-1}_{ij}\,. \tag{2.3.49}$$

Here the matrix L_{ij} incorporates the leading order dissipation and $L_{ij} \to 0$ when translation invariance is retained. While all of these DC conductivities are large, κ^{ij}_{DC} and also κ in (2.3.44) are parametrically smaller as pointed out in [45, 289]. Thus, from (2.3.46) we deduce that one of the frequencies will be proportional to L^{-1} while the other will be parametrically smaller.

Reduced hydrodynamics

When translations are broken, it should also be possible to construct a 'reduced' hydrodynamical description that just involves the conserved charges i.e. the heat and the $U(1)$ charge. At the level of linear response, this can be done, in principle, by solving for $\delta u_i^{(n)}$ order by order in the equations (2.3.23), to, effectively, get a set of linear equations for the variables δT and $\delta\mu$ and highly non-local in terms of the background metric and gauge-field. We will not carry out this in any detail here, but instead highlight some interesting features of the leading order terms that would arise. In particular, we will be able to derive a set of reduced hydrodynamical equations, at the level of linear response, that generalise those discussed in [231].

We begin with the on-shell expressions for the currents in the ε expansion given in (2.3.34). Focussing on the $U(1)$ current for the moment, we recall that at each order $\sqrt{h} f J^{i(n)}$ are periodic functions of the x^i. We have seen that at leading order they are determined by the system of linear equations given in (2.3.37), which is equivalent to the system of equations (2.3.28) that appeared for the calculation of the DC conductivity if we identify $\bar{E}_i \leftrightarrow -ik_i f\delta\mu^{(0)}$, $\bar{\zeta}_i \leftrightarrow -ik_j T_0^{-1}\delta T^{(0)}$. We can therefore write $\sqrt{h} f J^{i(1)}$ linearly in terms of $f\delta\mu^{(0)}$, $f\delta T^{(0)}$ as a sum of a constant flux, expressed in terms of the DC conductivity matrix, and a term which is co-closed and has vanishing zero mode (a periodic magnetisation current). Thus, we can write for the full current

$$\begin{aligned}\sqrt{h} f J^i = e^{-i\omega t} e^{i\varepsilon k_i x^i}\varepsilon\Big[&(\sigma^{ij}_{DC} + \partial_k S^{kij})(-ik_j\, f\delta\mu^{(0)}) \\ &+ (\alpha^{ij}_{DC} + \partial_k A^{kij})(-ik_j f\delta T^{(0)}) + \mathcal{O}(\varepsilon)\Big] ,\end{aligned} \tag{2.3.50}$$

where $S^{kij} = -S^{ikj}$, $A^{kij} = -A^{ikj}$ and both are periodic functions of the spatial coordinates. We can also obtain a similar expression for the heat current and we can write both of them in the following suggestive form

$$\begin{aligned}\sqrt{h} f J^i &= -(\sigma^{ij}_{DC} + \partial_k S^{kij})\nabla_j\delta\hat{\mu} - (\alpha^{ij}_{DC} + \partial_k A^{kij})\nabla_j\delta\hat{T} + \dots ,\\ \sqrt{h} f Q^i &= -\bar{T}_0(\alpha^{ij}_{DC} + \partial_k A^{kij})\nabla_j\delta\hat{\mu} - (\bar{\kappa}^{ij}_{DC} + \partial_k K^{kij})\nabla_j\delta\hat{T} + \dots ,\end{aligned} \tag{2.3.51}$$

where $\delta\hat{\mu} \equiv e^{-i\omega t} e^{i\varepsilon k_i x^i} f\delta\mu^{(0)}$, $\delta\hat{T} \equiv e^{-i\omega t} e^{i\varepsilon k_i x^i} f\delta T^{(0)}$ and $K^{kij} = -K^{ikj}$. In these on-shell expressions ω is fixed as an expansion in ε in terms of k_i and the background quantities via the dispersion relations.

We next consider analogous expressions for the local charge density and heat density. From (2.3.20) we obtain

$$\begin{aligned}\sqrt{h} f J^t &= \sqrt{h}\rho_0 + e^{-i\omega t} e^{i\varepsilon k_i x^i}\sqrt{h}\left[\xi_0\delta T^{(0)} + \chi_0\delta\mu^{(0)} + \mathcal{O}(\varepsilon)\right] ,\\ \sqrt{h} f Q^t &= \sqrt{h} f(\epsilon_0 - \mu_0\rho_0) + e^{-i\omega t} e^{i\varepsilon k_i x^i}\sqrt{h} f\left[c_{\mu 0}\delta T^{(0)} + T_0\xi_0\delta\mu^{(0)} + \mathcal{O}(\varepsilon)\right] ,\end{aligned} \tag{2.3.52}$$

where $\sqrt{h}\rho_0$ and $\sqrt{h} f(\epsilon_0 - \mu_0\rho_0)$ are the local charge densities in equilibrium. Hence, for the perturbation we can write

$$\begin{aligned}\delta[\sqrt{h} f J^t] &= \sqrt{h} f^{-1}\xi_0\delta\hat{T} + \sqrt{h} f^{-1}\chi_0\delta\hat{\mu} + \dots ,\\ \delta[\sqrt{h} f Q^t] &= \sqrt{h} c_{\mu 0}\delta\hat{T} + T_0\sqrt{h}\xi_0\delta\hat{\mu} + \dots .\end{aligned} \tag{2.3.53}$$

At this stage, from these on-shell expressions, we now can see the leading order structure of an off-shell reduced hydrodynamics. Specifically, if we take (2.3.53) to be expressions for the local charge densities and (2.3.51) to be the associated constitutive relations for the currents, the continuity equations $\nabla_\mu J^\mu = \nabla_\mu Q^\mu = 0$ at order ε^2 will lead to the same diffusive solutions that we had above with exactly the same dispersion relations for the diffusion modes. In particular, the magnetisation currents in (2.3.51) do not play a role in this specific calculation. It is also worth emphasising that in this reduced hydrodynamics, the variables $\delta\hat{T}$, $\delta\hat{\mu}$ need not be periodic functions and indeed they are not in the diffusive solutions.

We can now compare these results with the hydrodynamics described in the "Methods" section of [231], highlighting several differences. Firstly, the constitutive relations for the local currents given in [231] were declared to be given in terms of the DC conductivity, whereas here we have derived them from the underlying relativistic hydrodynamics. Secondly, the possibility of the terms involving S^{kij}, A^{kij}, K^{kij} was not considered in [231]. Finally, the expression for the local charge densities in [231] were not of the form (2.3.53). To make a connection we note that using (2.3.15), (2.3.16) we can rewrite (2.3.53) in the form

$$\begin{aligned}\delta[\sqrt{h}fJ^t] &= \left(\frac{\partial\rho_{tot}}{\partial\bar{T}_0}+\dots\right)\delta\hat{T}+\left(\frac{\partial\rho_{tot}}{\partial\bar{\mu}_0}+\dots\right)\delta\hat{\mu}+\dots\,,\\ \delta[\sqrt{h}fQ^t] &= \left(\bar{T}_0\frac{\partial s_{tot}}{\partial\bar{T}_0}+\dots\right)\delta\hat{T}+\left(\bar{T}_0\frac{\partial s_{tot}}{\partial\bar{\mu}_0}+\dots\right)\delta\hat{\mu}+\dots\,.\end{aligned} \tag{2.3.54}$$

where in the bracketed terms we have just written the constant zero mode part of the relevant term. The expressions (2.3.54) are what were considered in [231]; while the neglected higher Fourier modes will not affect the calculation of the dispersion relations for the diffusive modes, they are the same order in the ε expansion with the zero modes and they should be included as they will affect other calculations.

Green's functions

Within the context of relativistic hydrodynamics, the leading order solutions for the charge density and the currents are given in the previous subsection. It is possible to relate these expressions to the retarded Green's functions. At a first pass this seems problematic as the diffusive solutions are source free solutions and yet to extract Green's functions we need to relate a response to a source.

This puzzle can be resolved by the following trick. We view the solutions as having arisen after adiabatically switching on sources for the charge density in the far past, switching them off at time $t=0$ and then comparing the solutions for $t>0$ in the long wavelength limit. As this is somewhat technical we have explained how this can be achieved, as well as presenting some results of general validity, in appendix 2.B. For simplicity, we will carry out the analysis just for the case when there is only a single current present, which is the heat current. Hence, for convenience we present the perturbed part of the diffusive solution in this case here:

$$\begin{aligned}\delta[\sqrt{h}fQ^t] &= e^{-i\omega t}e^{i\varepsilon k_i x^i}\left[\sqrt{h}c_{\mu 0}f\delta T^{(0)}+\mathcal{O}(\varepsilon)\right]\,,\\ \sqrt{h}fQ^i &= e^{-i\omega t}e^{i\varepsilon k_i x^i}\varepsilon\left[(\bar{\kappa}^{ij}_{DC}+\partial_k K^{kij})(-ik_j)f\delta T^{(0)}+\mathcal{O}(\varepsilon)\right]\,,\end{aligned} \tag{2.3.55}$$

with $i\omega=\frac{\kappa^{ij}_{DC}k_ik_j}{\bar{T}_0\frac{\partial s_{tot}}{\partial\bar{T}_0}}$. We also recall that $f\delta T^{(0)}$ is constant and $\sqrt{h}c_{\mu 0}$ is a local susceptibility whose constant zero mode piece is $\bar{T}_0 \frac{\partial s_{tot}}{\partial T_0}$.

2.4 Final comments

In this paper we have made a general study of the hydrodynamical diffusion modes associated with conserved charges that arise in inhomogeneous media with a lattice symmetry. When the DC conductivities are finite, we showed that there are diffusive modes with dispersion relations that are determined by the DC conductivities and certain thermodynamical susceptibilities. This constitutes a generalised Einstein relation for inhomogeneous media. We also illustrated the general results, obtained by an analysis of retarded Greens functions, by considering the specific context of relativistic hydrodynamics. For simplicity, here we have focused on systems that are invariant under time reversal. However, it should be straightforward to generalise to the non-static case, after identifying suitably defined transport currents as in [132, 236, 291, 293, 312].

In [300], for a general conformal field theory on a curved manifold with a metric of the form (2.3.1) with $f = 1$, $h_{ij} = \Phi\delta_{ij}$ and Φ a periodic function, the relativistic hydrodynamic equations (with vanishing $U(1)$ fields) were solved for the local temperature and heat current, at the level of linear response, after applying a DC thermal gradient $\tilde{\zeta}_i$. In particular, it was shown that thermal backflow can occur whereby the heat current is locally flowing in the opposite direction to the DC source. These results can be recast in terms of the diffusion results of this paper. Let $\omega^{(2)}$ be the leading order dispersion relation as in (2.3.47). Then, focussing on real variables, we have leading order diffusing solutions with $\delta T = e^{-\varepsilon^2\omega^{(2)}t}\cos(\varepsilon k_i x^i)(\delta T^{(0)} + \varepsilon\delta T^{(1)} + \mathcal{O}(\varepsilon^2))$, and the local heat current given by $\delta Q^i = e^{-\varepsilon^2\omega^{(2)}t}\sin(\varepsilon k_i x^i)\varepsilon(\delta Q^{i(1)} + \mathcal{O}(\varepsilon))$, where $\delta T^{(1)}$ and $\delta Q^{i(1)}$ are the local temperature and heat current obtained in [300] for a DC thermal gradient given by $\tilde{\zeta}_i = k_i\delta T^{(0)}$. We can consider these solutions as having been adiabatically prepared in an initial state at $t = 0$ (say) and then diffusing. The solution shows that in each individual spatial period there is an elaborate local structure, which includes thermal backflow, with an overall damping of the current in time.

The existence of the same backflow current patterns that emerge in the steady state set-up provides a non-trivial test of the validity of hydrodynamics for certain strongly correlated systems of electrons for which backflows have been observed. Finally, we note that the initial conditions at $t = 0$ that we are considering, arising from the construction of specific long wavelength diffusion modes, might seem fine tuned. However, as long as short wavelength modes die out faster in time, the diffusive modes will capture the universal late time behaviour for generic initial conditions. For systems with light, spatially modulated modes, this will be case provided we examine long enough wavelengths.

The general results of this paper should also manifest themselves within the context of holography. In particular, it should be possible to obtain the Einstein relations in terms of the DC conductivities and the thermodynamic susceptibilities. It is now understood how, in general, the thermoelectric DC conductivity of the boundary field theory, when finite, can be obtained in terms of data on the black hole horizon [45, 290, 293]. Thus, providing one can obtain the susceptibilities in terms of horizon data, one should also be able to extract the Einstein relations. This will be explored in [2]. This line of investigation could also make contact with the recent work on relating diffusion to a characteristic velocity extracted from the black hole horizon, related to out of time ordered correlators [103, 104] and [105, 106, 299, 313–319].

2.A General results

Here we present some general results for Green's functions involving a single conserved current density operator J^μ satisfying the continuity equation $\partial_\mu J^\mu = 0$. We will present results for $G_{J^\mu J^\nu}(\omega, \mathbf{k}, \mathbf{k}')$; using the crystallographic decomposition (2.2.7) we can easily extract analogous results for the $G^{(\{n_j\})}_{J^\mu J^\nu}(\omega, \mathbf{k})$.

From (2.2.6) the current conservation condition $\partial_\mu J^\mu = 0$ implies

$$-i\omega\, G_{J^t B}(\omega, \mathbf{k}, \mathbf{k}') + i\mathbf{k}_i G_{J^i B}(\omega, \mathbf{k}, \mathbf{k}') = 0\,, \tag{2.A.1}$$

for any operator B, whose equal time commutator with J^t vanishes. From (2.A.1) we have

$$\begin{aligned} -i\omega\, G_{J^t J^t}(\omega, \mathbf{k}, \mathbf{k}') + i\mathbf{k}_i G_{J^i J^t}(\omega, \mathbf{k}, \mathbf{k}') &= 0\,,\\ -i\omega\, G_{J^t J^j}(\omega, \mathbf{k}, \mathbf{k}') + i\mathbf{k}_i G_{J^i J^j}(\omega, \mathbf{k}, \mathbf{k}') &= 0\,. \end{aligned} \tag{2.A.2}$$

We next consider the time reversal invariance conditions (2.2.17). Since $\epsilon_{J^t} = +1$ and $\epsilon_{J^i} = -1$, we obtain

$$\begin{aligned} G_{J^i J^t}(\omega, \mathbf{k}, \mathbf{k}') &= -G_{J^t J^i}(\omega, -\mathbf{k}', -\mathbf{k})\,,\\ G_{J^i J^j}(\omega, \mathbf{k}, \mathbf{k}') &= G_{J^j J^i}(\omega, -\mathbf{k}', -\mathbf{k})\,. \end{aligned} \tag{2.A.3}$$

Combing (2.A.3) with (2.A.2) we therefore have

$$\begin{aligned} \mathbf{k}_i \mathbf{k}'_j\, G_{J^i J^j}(\omega, \mathbf{k}, \mathbf{k}') &= -(i\omega)^2\, G_{J^t J^t}(\omega, \mathbf{k}, \mathbf{k}')\,,\\ i\mathbf{k}'_j\, G_{J^i J^j}(\omega, \mathbf{k}, \mathbf{k}') &= (i\omega)\, G_{J^i J^t}(\omega, \mathbf{k}, \mathbf{k}')\,. \end{aligned} \tag{2.A.4}$$

Define the static susceptibility

$$\lim_{\omega\to 0+i0} G_{J^tJ^t}(\omega,\mathbf{k},\mathbf{k}') \equiv -\chi_{J^tJ^t}(\mathbf{k},\mathbf{k}')\,. \tag{2.A.5}$$

We see that (2.A.2) and (2.A.4) imply

$$\begin{aligned}\mathbf{k}_i\chi_{J^iJ^t}(\mathbf{k},\mathbf{k}') &= 0\,,\\ \mathbf{k}_i\mathbf{k}'_j\chi_{J^iJ^j}(\mathbf{k},\mathbf{k}') &= 0\,.\end{aligned} \tag{2.A.6}$$

Note that the sign in (2.A.5) is fixed as follows. From (2.2.5), for a time-independent source for the charge density $\delta h_{J^t}(\mathbf{x})$, we have

$$\delta\langle J^t\rangle(t,\mathbf{k}) = \frac{1}{(2\pi)^d}\int d\mathbf{k}' G_{J^tJ^t}(\omega=0,\mathbf{k},\mathbf{k}')\,\delta h_{J^t}(\mathbf{k}')\,. \tag{2.A.7}$$

On the other hand from (2.2.4) $\delta H = (2\pi)^{-d}\int d\mathbf{k}\delta h_{J^t}(-\mathbf{k})\delta J^t(\mathbf{k})$ and so we identify the perturbed chemical potential, $\delta\mu(\mathbf{k})$, as $\delta\mu(\mathbf{k}) = -\delta h_{J^t}(\mathbf{k})$. Since the static susceptibility $\chi_{J^tJ^t}$ is defined by varying the charge density with respect to the chemical potential we get the sign as in (2.A.5).

2.B Linear response from a prepared source

We consider a perturbative deformation of the Hamiltonian as in (2.2.4), with a prepared source that is switched off at $t=0$, given by

$$h_B(t,\mathbf{x}) = \begin{cases} e^{\varepsilon_t\, t + i\,\mathbf{k}_s\,\mathbf{x}}\,\delta h_B, & t\le 0\\ 0 & t>0\end{cases}\,, \tag{2.B.1}$$

with $\varepsilon_t > 0$. This source contains a single spatial Fourier mode and we will be interested in taking the adiabatic limit $\varepsilon_t \to 0^+$.

The time dependent expectation value of an operator A is given by the retarded Green's function as in (2.2.5). Thus, at $t=0$, when the sources are switched off, we have

$$\begin{aligned}\delta\langle A\rangle(t=0,\mathbf{x}) &= \int dt'\,d\mathbf{x}'\,G_{AB}(-t',\mathbf{x},\mathbf{x}')\,\delta h_B(t',\mathbf{x}')\,,\\ &= \int dt'\,d\mathbf{x}'\,G_{AB}(t',\mathbf{x},\mathbf{x}')\,e^{-\varepsilon_t t' + i\,\mathbf{k}_s\mathbf{x}'}\,\delta h_B\,,\\ &= G_{AB}(i\,\varepsilon_t,\mathbf{x},\mathbf{k}_s)\,\delta h_B\,.\end{aligned} \tag{2.B.2}$$

In the $\varepsilon_t \to 0^+$ limit, after a Fourier transform, we have

$$\delta\langle A\rangle(t=0,\mathbf{k}) = -\chi_{AB}(\mathbf{k},\mathbf{k}_s)\,\delta h_B\,, \tag{2.B.3}$$

where $\chi_{AB}(\mathbf{k},\mathbf{k}') \equiv -\lim_{\omega\to 0+i0} G_{AB}(\omega,\mathbf{k},\mathbf{k}')$. Also, after a Fourier transform of the source (2.B.1), for any $t>0$ we deduce that

$$\delta\langle A\rangle(t,\mathbf{x}) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\omega\, \frac{1}{\varepsilon_t + i\,\omega} e^{-i\omega\, t} G_{AB}(\omega,\mathbf{x},\mathbf{k}_s)\,\delta h_B\,. \tag{2.B.4}$$

Taking a Laplace transform in time we get

$$\begin{aligned}\delta\langle A\rangle(z,\mathbf{x}) &\equiv \int_0^{+\infty} dt\,\delta\langle A\rangle(t,\mathbf{x})\, e^{i\,zt}\,,\\ &= -\frac{1}{2\pi}\int_{-\infty}^{+\infty} d\omega\,\frac{1}{\omega - i\,\varepsilon_t}\frac{1}{\omega - z}\, G_{AB}(\omega,\mathbf{x},\mathbf{k}_s)\,\delta h_B\,,\end{aligned} \tag{2.B.5}$$

with, necessarily, $\mathrm{Im}\, z > 0$ in order for the integrals to converge. Performing a contour integral on the above expression by closing it in the upper half plane and assuming that the Green's function vanishes fast enough for large ω, we just pick up contributions from the poles at $\omega = i\,\varepsilon_t$ and $\omega = z$ to obtain

$$\delta\langle A\rangle(z,\mathbf{x}) = -\frac{i}{i\,\varepsilon_t - z} G_{AB}(i\,\varepsilon_t,\mathbf{x},\mathbf{k}_s)\,\delta h_B - \frac{i}{z - i\,\varepsilon_t} G_{AB}(z,\mathbf{x},\mathbf{k}_s)\,\delta h_B\,. \tag{2.B.6}$$

Thus, in the $\varepsilon_t \to 0^+$ limit we conclude that the spatial Fourier transform is given by (2.B.1):

$$\delta\langle A\rangle(z,\mathbf{k}) = \frac{1}{i\,z}\left(G_{AB}(z,\mathbf{k},\mathbf{k}_s) + \chi_{AB}(\mathbf{k},\mathbf{k}_s)\right)\,\delta h_B\,. \tag{2.B.7}$$

Using (2.B.3) we now obtain the following solution to the initial value problem that is sourced by (2.B.1) in the $\varepsilon_t \to 0^+$ limit:

$$\delta\langle A\rangle(z,\mathbf{k}) = -\frac{1}{i\,z}\left(G_{AB}(z,\mathbf{k},\mathbf{k}_s)\chi^{-1}_{BC}(\mathbf{k},\mathbf{k}_s) + \delta_{AC}\right)\delta\langle C\rangle(t=0,\mathbf{k})\,. \tag{2.B.8}$$

2.B.1 Conserved current

Let us now apply some of these results to conserved currents. For simplicity we just consider the case of a single conserved current and assume that there is a single diffusion pole. We will assume that the source (2.B.1) is a source just for the charge density operator J^t. In particular at $t=0$ we write the source as $e^{i\mathbf{k}_s\mathbf{x}}\delta h^{(0)}_{J^t}$, with constant $\delta h^{(0)}_{J^t}$. We take the limit $\varepsilon_t \to 0$ and then consider $\mathbf{k}_s \to 0$.

From (2.B.6), the time dependence of the charge density for $t>0$ is fixed by the Laplace transformed quantity

$$\delta\langle J^t\rangle(z,\mathbf{x}) = e^{i\,\mathbf{k}_s\mathbf{x}} \sum_{\{n_j\}} e^{i n_j \mathbf{k}^j_L \mathbf{x}} \frac{1}{iz}\left[G^{(\{n_j\})}_{J^tJ^t}(z,\mathbf{k}_s) + \chi^{(\{n_j\})}_{J^tJ^t}(\mathbf{k}_s)\right]\frac{1}{(2\pi)^d}\delta h^{(0)}_{J^t}\,, \tag{2.B.9}$$

where $\chi^{(\{n_j\})}_{J^tJ^t}(\mathbf{k}_s) = -\lim_{\omega\to 0+i0} G^{(\{n_j\})}_{J^tJ^t}(\omega,\mathbf{k}_s)$. It is interesting to now examine the zero mode of the periodic function inside the sum (see (2.2.19)):

$$\delta\langle J^t\rangle^{\{(0)\}}(z) = \frac{1}{iz}\left[G_{J^tJ^t}(z,\mathbf{k}_s) + \chi_{J^tJ^t}(\mathbf{k}_s)\right]\frac{1}{(2\pi)^d}\delta h^{(0)}_{J^t}\,, \tag{2.B.10}$$

since we can draw some further general conclusions using the results of section 2.2.1. Indeed after considering $\mathbf{k}_s \to 0$, and recalling the general results (2.2.30) and (2.2.33), we have

$$\delta\langle J^t\rangle^{\{(0)\}}(z) = \frac{-1}{-iz + \mathbf{k}_{si}\mathbf{k}_{sj}\sigma^{ij}(z)\chi(\mathbf{0})^{-1}}\chi(\mathbf{0})\frac{1}{(2\pi)^d}\delta h^{(0)}_{J^t}\,. \tag{2.B.11}$$

Taking the inverse Laplace transform and keeping just the time-dependence that is leading order in $\mathbf{k}_s$, we obtain

$$\delta\langle J^t\rangle^{\{(0)\}}(t) = -e^{-i\omega(\mathbf{k}_s)t}\chi(\mathbf{0})\frac{1}{(2\pi)^d}\delta h^{(0)}_{J^t}\,. \tag{2.B.12}$$

with $i\omega(\mathbf{k}_s) = \sigma^{ij}_{DC}\mathbf{k}_{si}\mathbf{k}_{sj}\chi(\mathbf{0})^{-1}$.

We can now make a comparison with the diffusive solutions given in (2.3.55) that we found within the context of relativistic hydrodynamics. Recalling that in this appendix, and also in section 2.2, we are considering current densities, whereas in section 2.3 we used current vectors, we therefore should compare the local current $\delta[\sqrt{h}fQ^t(t,\mathbf{x})]$ in (2.3.55) with $\delta\langle J^t\rangle(t,\mathbf{x})$. Identifying the constant source $\frac{1}{(2\pi)^d}\delta h^{(0)}_{J^t}$ here with $-f\delta T^{(0)}$ (see the discussion following (2.A.7)), after comparing (2.3.55) with (2.B.9) and the above analysis, we conclude that for these particular solutions we have that for each $\{n_j\}$, in the limit that $\mathbf{k}_s \to 0$,

$$G^{(\{n_j\})}_{J^tJ^t}(\omega,\mathbf{k}_s)\chi^{(\{n_j\})}_{J^tJ^t}(\mathbf{k}_s)^{-1} + 1 \to \frac{1}{-i\omega + \mathbf{k}_{si}\mathbf{k}_{sj}S^{\{n_j\}ij}(\omega)\chi(\mathbf{0})^{-1}}\,, \tag{2.B.13}$$

with $S^{\{n_j\}ij(\omega)} = \sigma^{ij}_{DC} + \mathcal{O}(\omega)$, in order to get the correct time-dependence. In particular, all of these modes of the Green's function have the same diffusion pole at the origin.

We next consider the spatial components of the current. Starting with (2.B.8) and using (2.A.4) we can write

$$\delta\langle J^i\rangle(z,\mathbf{k}) = \left[\frac{1}{(iz)^2}G_{J^iJ^j}(z,\mathbf{k},\mathbf{k}_s)(-i\mathbf{k}_{sj}) + \frac{1}{iz}\chi_{J^iJ^t}(\mathbf{k},\mathbf{k}_s)\right]\delta h^{(0)}_{J^t}\,. \tag{2.B.14}$$

After a Fourier transform on the spatial coordinates we can therefore write

$$\begin{aligned}\delta\langle J^i\rangle(z,\mathbf{x}) = e^{i\,\mathbf{k}_s\mathbf{x}}\sum_{\{n_j\}} e^{in_j\mathbf{k}^j_L\mathbf{x}}\Big[&\frac{1}{(iz)^2}G^{(\{n_j\})}_{J^iJ^j}(z,\mathbf{k}_s)(-i\mathbf{k}_{sj})\\ &+\frac{1}{iz}\chi^{(\{n_j\})}_{J^iJ^t}(\mathbf{k}_s)\Big]\frac{1}{(2\pi)^d}\delta h^{(0)}_{J^t}\,.\end{aligned} \tag{2.B.15}$$

Current conservation implies that $\mathbf{k}_i \chi_{J^i J^t}(\mathbf{k}, \mathbf{k}_s) = 0$ (see (2.A.6)) but in general $\chi_{J^i J^t}(\mathbf{k}, \mathbf{k}_s) \neq 0$. However, in the relativistic hydrodynamics in the static background we do have $\chi_{J^i J^t}(\mathbf{k}, \mathbf{k}_s) = 0$ (see the comment below (2.3.12)). Thus, comparing (2.B.15) with (2.3.55) we deduce that for the relativistic hydrodynamics, as $\mathbf{k}_s \to 0$ we have

$$\frac{1}{(i\omega)^2} G^{(\{n_j\})}_{J^i J^j}(\omega, \mathbf{k}_s)(-i\mathbf{k}_{sj}) \to \frac{(\bar{\kappa}^{ij}_{DC} + \partial_k K^{kij})^{(\{n_j\})}(-i\mathbf{k}_{sj})}{-i\omega + \mathbf{k}_{si}\mathbf{k}_{sj}\tilde{S}^{\{n_j\}ij}(\omega)\chi(\mathbf{0})^{-1}} \,, \tag{2.B.16}$$

with $\tilde{S}^{\{n_j\}ij(\omega)} = \sigma^{ij}_{DC} + \mathcal{O}(\omega)$ in order to get the correct time-dependence.

A final comment is that if we consider (2.B.3) with $\chi_{J^i J^t}(\mathbf{k}, \mathbf{k}_s) = 0$ then we deduce that $\delta\langle J^i \rangle(t = 0, \mathbf{x}) = 0$. This seems inconsistent with the $t = 0$ limit of the diffusive solution arising from hydrodynamics. The resolution of this puzzle is that when we take the limit $\varepsilon_t \to 0$ it leads to a discontinuity in the current. The correct thing to do is compare the currents for $t > 0$ as we did above.

Chapter 3

Diffusion for holographic lattices

This chapter is a reproduction of [2], written in collaboration with Aristomenis Donos and Jerome Gauntlett.

Having studied thermoelectric diffusion in QFTs in chapter 2, we now turn our attention to holographic field theories. We consider black hole spacetimes that are holographically dual to strongly coupled field theories in which spatial translations are broken explicitly. As in the DC conductivities prescription of [45, 290], we write down an ansatz for a static, planar and periodic spacetime, without specifying an exact solution (3.2.2). This implies that our results are universal within the class of EMD theories (3.2.1), for solutions describing systems with strong momentum relaxation.

In section 3.2 we discuss details of such backgrounds, and we also express the thermodynamic susceptibilities in terms of horizon data. At first glance, this seems to contradict the common belief that this is only possible in special cases, with the susceptibilities normally involving integrals over the whole bulk. The expressions we present involve only integrals over the horizon; however, they also involve variations with respect to boundary data (the chemical potential $\bar{\mu}$). Essentially, we are just trading knowledge of a full bulk solution with knowledge of a family of horizons.

In section 3.3 we derive the horizon constraints that a general quasinormal mode should satisfy. We then proceed in section 3.4 to discuss how the quasinormal modes associated with diffusion of heat and charge can be systematically constructed in a long wavelength perturbative expansion. We show that the dispersion relation for these modes is given in terms of the thermoelectric DC conductivity and static susceptibilities of the dual field theory and thus we derive a generalised Einstein relation from Einstein's equations, as in chapter 2. It is particularly satisfying that the dissipative part of the diffusion constants, i.e. the DC conductivities, are given by solving a system of generalised Navier-Stokes equations on the black hole horizon, realising in yet another instance the dissipative nature of horizons.

Finally, as a corollary of our results we find that thermodynamic instabilities imply specific types of dynamical instabilities of the associated black hole solutions.

3.1 Introduction

Holography provides a powerful theoretical framework for studying the properties of strongly coupled quantum critical systems. A basic feature is that a given quantum system in thermal equilibrium is described by a stationary black hole spacetime with a Killing horizon and, furthermore, the entropy and conserved charges can be universally determined by data on the black hole horizon (e.g. see [40] and references therein). Going beyond thermal equilibrium and moving to the realm of linear response, it has been shown that the thermoelectric DC conductivity, when finite, is also universally determined by data on the horizon, by solving a specific Stokes flow for an auxiliary fluid on the horizon [290] (further extensions are discussed in [45, 291, 293, 320]).

It is natural to enquire if other properties of the dual field theory can also be obtained from horizon data. In this paper we discuss the construction of quasi-normal modes that are dual to the long wave-length hydrodynamic modes associated with diffusion of heat and electric charge. In particular, we will show how the dispersion relation for these modes can also be obtained in terms of the properties of the black hole solutions at the horizon.

Recall that in the specific context of translationally invariant and charge neutral systems the diffusion of electric charge was first discussed some time ago in [185]. Furthermore, again for this specific setup, an Einstein relation, relating the associated electric diffusion constant to the finite DC conductivity and static charge susceptibilities, was derived in [85], where it was also shown how the DC conductivity can be obtained explicitly from the horizon[1]. It should be noted that in this set up the thermal DC conductivity is infinite and, correspondingly, there is no heat diffusion mode. In this paper we will discuss the diffusion of both electric charge and heat within the general context of charged and spatially inhomogeneous media. The spatial inhomogeneities that we consider arise from breaking of spatial translations explicitly, and the black holes are known as ‘holographic lattices’ [216].

In a recent paper [1] we carried out an analysis of the diffusion of conserved charges in the context of spatially inhomogeneous media for arbitrary quantum field theories (not necessarily holographic). Subject to the retarded current-current correlators satisfying some general analyticity conditions, as well as assuming that

[1]From the universal perspective of [290], this is a special set up where the Stokes flow equations are solved trivially.

the thermoelectric DC conductivity is finite, the long wavelength hydrodynamical modes associated with diffusion of charge and heat were identified and a generalised Einstein relation was derived. In addition, the general formalism was illustrated for thermoelectric diffusion within the context of relativistic hydrodynamics where momentum dissipation was achieved not by modifying the conservation equations, as is usually done, but by explicitly breaking translations by considering the system with spatially modulated sources for the stress tensor and electric current as in [300, 304].

Given the general results presented in [1], one anticipates that it should be possible to derive the generalised Einstein relations within the context of arbitrary holographic lattices. Since, for this case, the DC conductivity is finite and is equal to a horizon DC conductivity that is obtained by solving a Stokes flow on the horizon, one can ask about the relevant charge susceptibilities. Since the conserved charges can be evaluated at the horizon provided one knows how this data depends on changing the temperature and the chemical potential of the black holes in thermal equilibrium, one can also obtain horizon expressions for the susceptibilities. As we will see, this simple observation about the susceptibilities will be sufficient to extract the dispersion relations for the diffusive modes and hence the Einstein relation. In slightly more detail, using a radial decomposition of the equations of motion, we will explain how the quasi-normal diffusion modes can be systematically constructed in a long wavelength, perturbative expansion. In general, while both the radial equations and the constraint equations are required to carry out this construction, we will see that an analysis of just the constraint equations on the horizon are sufficient to extract the Einstein relation, which is the universal part of the dispersion relation in the long wavelength expansion for the diffusive modes.

Recently there has been a particular focus on studying diffusion of heat and charge in the context of holography. This stems, in part, from the suggestion that diffusive processes may be a key to understanding universal aspects of transport in incoherent metals [231]. Furthermore, it was also suggested in [231] that there might be lower bounds on diffusion constants by analogy with bounds on shear viscosity associated with diffusion of momentum [198]. A key idea is to write $D \sim v^2\tau$, where D is suitable diffusion constant and v, τ are characteristic velocities and time scales of the system, and it was suggested in [231] that τ should be the 'Planckian time scale' $\tau = \hbar/(k_B T)$ [321, 322]. An interesting subsequent development was the suggestion that v should be identified with the butterfly velocity, v_B, extracted from out of time order correlators [103, 104] and used as a measure of the onset of quantum chaos.

While there has been a range of interesting holographic results in this direction, including [105–107, 299, 313–319, 323–325], with an appreciation that it is the thermal diffusion should be related to v_B, it is fair to say that within holography a sharp global picture has yet to emerge[2]. Almost all of the holographic study in this area has been in the setting of specific types of 'homogeneous' holographic lattices [278, 279], which maintain a translationally invariant metric. In these cases it is straightforward to extract v_B by studying a shock wave entering the black hole horizon [98, 102] (see also [334]). A notable exception is [299] who studied holographic lattices in one spatial dimension, but working in a hydrodynamic, high temperature limit of the background holographic lattice. We hope that the present work, which illuminates universal aspects of diffusion for arbitrary spatial modulation in holography, will be useful in further developments.

In a different direction, our derivation of the dispersion relation leads to a general connection between thermodynamic instabilities and dynamic instabilities. Some time ago, building on [257, 258], it was shown in a specific holographic context with a translationally invariant horizon, that thermodynamic instability implies an imaginary speed of sound, leading to unstable quasi-normal modes and dynamical instability[3] [339]. For general spatial modulation within holography, any sound modes will only appear on scales much smaller than the scale of the modulation and hence this will not be a universal channel to deduce dynamical instability from thermodynamic instability. Instead, the diffusion modes do provide such a channel. Specifically, in the presence of spatial modulation, we can deduce the following result. If the heat and charge susceptibility matrix has a negative eigenvalue, then the system is thermodynamically unstable and then the dispersion relation implies that there is at least one mixed diffusion mode, involving heat and charge, living in the upper half plane which will necessarily lead to a dynamical instability.

3.2 Background black hole solutions

We will consider a general class of bulk theories which couple the metric to a gauge field A_μ, with field strength $F_{\mu\nu}$, and a scalar field ϕ in D spacetime dimensions, governed by an action of the form

$$S = \int d^D x\sqrt{-g}\left(R - V(\phi) - \frac{Z(\phi)}{4}\,F^2 - \frac{1}{2}\,(\partial\phi)^2\right)\,. \tag{3.2.1}$$

The only constraints that we impose on the functions $V(\phi), Z(\phi)$ is that the equations of motion admit an AdS_D vacuum solution with $\phi = A_\mu = 0$. We assume that in

---

[2] It is striking that a relation of the form $D \sim v_B^2\tau$ has also appeared in a variety of other non-holographic contexts, including [314, 326–333], with $\tau \sim \lambda_L^{-1}$ where λ_L is the Lyapunov exponent [328].

[3] Some recent discussion of both hydrodynamic and non-hydrodynamic modes and the connection with instabilities in a translationally invariant setting, appeared in [335–338].

this vacuum the scalar field ϕ is dual to an operator with conformal dimension Δ. We have also set $16\pi G = 1$ for convenience.

We are interested in studying the family of static, background black hole solutions that lie within the ansatz

$$ds^2 = -UG\,dt^2 + \frac{F}{U}\,dr^2 + ds^2(\Sigma_d)\,,$$
$$A = a_t dt\,, \tag{3.2.2}$$

with $ds^2(\Sigma_d) = g_{ij}dx^i dx^j$ and $d = D-2$. The functions G, F, a_t, ϕ and the metric components g_{ij} are all independent of the time coordinate t and depend on (r, x^i). Note that the function $U = U(r)$, which is redundant, is included to conveniently deal with some aspects of the asymptotic behaviour of the solution.

Although it is possible to be more general, to simplify the presentation we will assume that we have single black hole Killing horizon, located at $r = 0$, and that the coordinates (t, r, x^i) are globally defined outside the black hole all the way out to the AdS_D boundary which will be located at $r \to \infty$. In particular, this means that the radial foliation is globally defined up to a 'stretched horizon' located at some small radial distance outside the black hole and that the topology of the black hole horizon is Σ_d. Similarly, we will also assume Σ_d has planar topology and all functions appearing in (3.2.2) are assumed to be periodic in the spatial directions x^i with period L_i, corresponding to static, periodic deformations of the dual CFT. It will be useful to define $\oint = (\prod L_i)^{-1}\int dx^1 \dots dx^d$ which allows us to extract the zero mode of periodic functions.

Asymptotically, as $r \to \infty$, the solutions are taken to approach AdS_D with boundary conditions that explicitly break translation invariance:

$$U \to r^2, \qquad F \to 1, \qquad G \to G^{(\infty)}(x), \qquad g_{ij}(r,x) \to r^2 g^{(\infty)}_{ij}(x),$$
$$a_t(r,x) \to \mu(x), \qquad \phi(r,x) \to r^{\Delta-d-1}\phi^{(\infty)}(x)\,. \tag{3.2.3}$$

This corresponds to placing the dual CFT on a curved spacetime manifold with metric given by $ds^2 = -G^{(\infty)}(x)dt^2 + g^{(\infty)}_{ij}(x)dx^i dx^j$, having a spatially dependent chemical potential $\mu(x)$ and deforming by a spatially dependent source $\phi^{(\infty)}(x)$ for the operator dual to ϕ. It will be convenient to separate out the zero mode of $\mu(x)$ by defining

$$\mu(x) \equiv \bar{\mu} + \tilde{\mu}(x) \tag{3.2.4}$$

with constant $\bar{\mu}$ and $\oint \tilde{\mu}(x) = 0$.

The Killing horizon will, in general, be spatially modulated and we have the following expansions[4] near $r=0$

$$\begin{aligned} U\left(r\right) &= r\left(4\pi\, T + U^{(1)}\, r + \ldots\right), & G(r,x) &= 1 + G^{(1)}\left(x\right) r + \ldots\,, \\ F(r,x) &= 1 + F^{(1)}\left(x\right) r + \ldots\,, & g_{ij} &= g^{(0)}_{ij} + g^{(1)}_{ij} r + \ldots\,, \\ a_t(r,x) &= r\left(a_t^{(0)} + a_t^{(1)}\left(x\right) r + \ldots\right), & \phi &= \phi^{(0)}(x) + r\phi^{(1)}(x) + \ldots\,. \end{aligned} \tag{3.2.5}$$

After a Euclidean continuation, we find that the constant T is the Hawking temperature of the black hole and can be identified with the constant temperature of the dual field theory[5]. We also note that we can introduce an ingoing Eddington-Finkelstein-like coordinate

$$v_{EF} \equiv t + \frac{\ln r}{4\pi T}\,, \tag{3.2.6}$$

and that the metric is regular in the (v_{EF}, r, x^i) coordinates as $r \to 0$.

3.2.1 Susceptibilities from the horizon

It will be important in the sequel to be able to express certain thermodynamic susceptibilities in terms of data at the horizon. More precisely we can obtain the susceptibilities provided that we know the horizon data as a function of the temperature T and the zero mode of the chemical potential $\bar{\mu}$. We first recall that the total entropy density of the system, s, can be expressed as

$$s = 4\pi \oint_H \sqrt{g_{(0)}}\,, \tag{3.2.7}$$

where the subscript H emphasises that this is an integral evaluated at the black hole horizon. Similarly the total charge density, $\rho \equiv J^t$, can be expressed either as a boundary quantity or a horizon quantity via

$$\rho \equiv \oint_\infty \sqrt{-g} Z(\phi) F^{tr} = \oint_H \sqrt{g_{(0)}} Z^{(0)} a_t^{(0)}\,, \tag{3.2.8}$$

where the equality can be deduced from the gauge equation of motion.

Hence under a constant variation of the temperature, $T \to T + \delta T$, and zero mode of the chemical potential, $\bar{\mu} \to \bar{\mu} + \delta\bar{\mu}$ (see (3.2.4)), we have

$$\begin{aligned} \delta s &\equiv T^{-1} c_\mu\, \delta T + \xi\, \delta\bar{\mu}\,, \\ \delta\rho &\equiv \xi\, \delta T + \chi\, \delta\bar{\mu}\,, \end{aligned} \tag{3.2.9}$$

[4]We have chosen our coordinates so that possible functions $G^{(0)}(x) = F^{(0)}(x)$ are set to unity.

[5]Since the CFT is defined on the boundary metric (3.2.3) there is also a natural notion of a local temperature of the dual field theory given by $T(x) = [G^{(\infty)}(x)]^{-1/2} T$. Also note that for the case of CFTs we can carry out a Weyl transformation to set $G^{(\infty)}(x) = 1$, suitably taking into account the possibility of a conformal anomaly.

where the thermodynamic susceptibilities are given by[6]

$$
\begin{aligned}
T^{-1}c_\mu &= 4\pi \oint_H d^dx \frac{1}{2}\sqrt{g_{(0)}}\,(g^{(0)})^{ij}\,\frac{\partial g^{(0)}_{ij}}{\partial T}\,,\\
\xi &= 4\pi \oint_H d^dx \frac{1}{2}\sqrt{g_{(0)}}\,(g^{(0)})^{ij}\,\frac{\partial g^{(0)}_{ij}}{\partial \bar{\mu}}\,,\\
&= \oint_H d^dx\,\sqrt{g_{(0)}}\left(Z^{(0)}\frac{\partial a^{(0)}_t}{\partial T} + \partial_\phi Z^{(0)}\,a^{(0)}_t\,\frac{\partial\phi^{(0)}}{\partial T} + \frac{1}{2}Z^{(0)}\,a^{(0)}_t(g^{(0)})^{ij}\,\frac{\partial g^{(0)}_{ij}}{\partial T}\right)\,,\\
\chi &= \oint_H d^dx\,\sqrt{g_{(0)}}\left(Z^{(0)}\frac{\partial a^{(0)}_t}{\partial \bar{\mu}} + \partial_\phi Z^{(0)}\,a^{(0)}_t\,\frac{\partial\phi^{(0)}}{\partial \bar{\mu}} + \frac{1}{2}Z^{(0)}\,a^{(0)}_t(g^{(0)})^{ij}\,\frac{\partial g^{(0)}_{ij}}{\partial \bar{\mu}}\right)\,.
\end{aligned}
\tag{3.2.10}
$$

The equality of the two expressions for ξ at the horizon is not obvious. However, from a boundary perspective it is just a Maxwell relation that arises from the first law. To see this we recall that we can calculate the renormalised, total free energy density, w_{FE}, from the total on-shell action after adding suitable boundary terms. For the ensemble of interest we have $s = -\delta w_{FE}/\delta T$ and $\rho = -\delta w_{FE}/\delta\bar{\mu}$ and the result at the horizon follows. Note that $c_\mu \equiv T(\partial s/\partial T)_{\bar{\mu}}$. Later we will also need $c_\rho \equiv T(\partial s/\partial T)_\rho$ which can be written as

$$
c_\rho \equiv c_\mu - \frac{T\xi^2}{\chi}\,. \tag{3.2.11}
$$

To see this we use $\xi/\chi = -(\partial\bar{\mu}/\partial T)_\rho = (\partial s/\partial\rho)_T$, where the second equality is a Maxwell relation.

3.3 Time dependent perturbation and the constraints

Consider a general perturbation of the background black hole solution (3.2.2) given by $\delta P \equiv \{\delta g_{\mu\nu},\, \delta a_\mu,\, \delta\phi\}$, with all quantities functions of all of the bulk coordinates (t, r, x^i). We want to consider time-dependence of the form $e^{-i\omega t}$. It is convenient to write

$$
\delta P(t, r, x^i) = e^{-i\omega[t+S(r)]}\delta\hat{P}(r, x^i)\,, \tag{3.3.1}
$$

with $S(r) \to 0$ as $r \to \infty$ and, in order to ensure that the perturbation satisfies ingoing boundary conditions at the black hole horizon, $S(r) \to \frac{\ln r}{4\pi T} + S^{(1)}\,r + \cdots$ as $r \to 0$.

[6] In section 3 of [1] these quantities were denoted by capital letters: C_μ, Ξ and X.

With this in hand, and recalling the definition of v_{EF}, the ingoing Eddington-Finkelstein-like coordinate in (3.2.6), we demand that near $r=0$ the perturbation behaves as

$$\begin{aligned}
\delta g_{tt} &= e^{-i\omega v_{EF}}(4\pi T r)\left(\delta g_{tt}^{(0)}(x)+\mathcal{O}(r)\right), \quad \delta g_{rr} = e^{-i\omega v_{EF}}\frac{1}{4\pi T r}\left(\delta g_{rr}^{(0)}(x)+\mathcal{O}(r)\right),\\
\delta g_{ij} &= e^{-i\omega v_{EF}}\,\delta g_{ij}^{(0)}(x)+\mathcal{O}(r), \qquad\qquad \delta g_{tr} = e^{-i\omega v_{EF}}\,\delta g_{tr}^{(0)}(x)+\mathcal{O}(r),\\
\delta g_{ti} &= e^{-i\omega v_{EF}}\,(\delta g_{ti}^{(0)}(x)+r\,\delta g_{ti}^{(1)}(x)+\mathcal{O}(r^2)),\\
\delta g_{ri} &= e^{-i\omega v_{EF}}\frac{1}{4\pi T r}\left(\delta g_{ri}^{(0)}(x)+r\,\delta g_{ri}^{(1)}(x)+\mathcal{O}(r^2)\right), \qquad (3.3.2)
\end{aligned}$$

as well as

$$\begin{aligned}
\delta a_t &= e^{-i\omega v_{EF}}\,(\delta a_t^{(0)}(x)+r\,\delta a_t^{(1)}(x)+\mathcal{O}(r^2)),\\
\delta a_r &= e^{-i\omega v_{EF}}\frac{1}{4\pi T r}\left(\delta a_r^{(0)}(x)+r\,\delta a_r^{(1)}(x)+\mathcal{O}(r^2)\right),\\
\delta a_i &= e^{-i\omega v_{EF}}\,(\delta a_i^{(0)}(x)+\mathcal{O}(r)),\\
\delta\phi &= e^{-i\omega v_{EF}}\,(\delta\phi^{(0)}(x)+\mathcal{O}(r)), \qquad (3.3.3)
\end{aligned}$$

with

$$\begin{aligned}
-2\pi T(\delta g_{tt}^{(0)}+\delta g_{rr}^{(0)}) = -4\pi T\,\delta g_{rt}^{(0)} &\equiv p\,,\\
\delta g_{ti}^{(0)} = \delta g_{ri}^{(0)} &\equiv -v_i,\\
\delta a_r^{(0)} = \delta a_t^{(0)} &\equiv w\,. \qquad (3.3.4)
\end{aligned}$$

There is some residual gauge invariance for the perturbation at the horizon, maintaining the ingoing boundary conditions, which we discuss in appendix 3.A.

3.3.1 Constraints

Using a radial decomposition of the equations of motion one obtains a set of constraints that must be satisfied on a surface of constant r. We want to evaluate these constraints for the perturbed solution at the black hole horizon. More precisely we evaluate the constraints on a stretched horizon located at a small radial distance r away from the horizon and then take the limit as $r\to 0$. The calculations are a generalisation of the calculations that were carried out in [45, 290]. Here we will just state the final result but we have presented some details in appendix 3.B.

The combined set of constraints include two scalar equations and a vector equation. If we define

$$\begin{aligned}
Q_{(0)}^i &= 4\pi T\sqrt{g_{(0)}}v^i\,,\\
J_{(0)}^i &= \sqrt{g_{(0)}}g_{(0)}^{ij}Z^{(0)}\left(\partial_j w+a_t^{(0)}v_j+i\omega\delta a_j^{(0)}\right). \qquad (3.3.5)
\end{aligned}$$

then the two scalar equations are

$$\partial_i Q^i_{(0)} = i\omega T\Big(2\pi\sqrt{g_{(0)}}g^{ij}_{(0)}\delta g^{(0)}_{ij}\Big)\,, \tag{3.3.6}$$

and

$$\begin{aligned}\partial_i J^i_{(0)} = i\omega\sqrt{g_{(0)}}\Big[&Z^{(0)}\left(a^{(0)}_t\left(\delta g^{(0)}_{tt}+\frac{p}{4\pi T}\right)+\delta a^{(1)}_t-\frac{i\omega}{4\pi T}\left(\delta a^{(1)}_t-\delta a^{(1)}_r\right)\right)\\ &+Z^{(0)}\left(\frac{1}{2}a^{(0)}_t g^{ij}_{(0)}\delta g^{(0)}_{ij}+\frac{1}{4\pi T}v^i\partial_i a^{(0)}_t\right)+\partial_\phi Z^{(0)}a^{(0)}_t\,\delta\phi^{(0)}\Big]\,.\end{aligned} \tag{3.3.7}$$

Finally the vector equation can be written as

$$\begin{aligned}&-2\,\nabla^j\nabla_{(j}v_{i)}-Z^{(0)}a^{(0)}_t\left(\nabla_i w+i\omega\delta a^{(0)}_i\right)+\nabla_i\phi^{(0)}\left(v^j\nabla_j\phi^{(0)}-i\omega\delta\phi^{(0)}\right)\\ &+(1+\frac{i\omega}{4\pi T})\nabla_i p = i\omega\left(\delta g^{(1)}_{ti}-\frac{i\,\omega}{4\pi T}(\delta g^{(1)}_{ti}-\delta g^{(1)}_{ri})+g^{(1)}_{il}v^l-\partial_i\delta g^{(0)}_{tt}-\nabla^k\delta g^{(0)}_{ki}\right)\,.\end{aligned} \tag{3.3.8}$$

where the covariant derivative is with respect to the horizon metric $g^{(0)}_{ij}$, which is also used to raise and lower indices. One can check that these equations are consistent with the residual gauge transformations mentioned above and are given explicitly in appendix 3.A.

Notice that if we set the frequency $\omega = 0$ in (3.3.6)-(3.3.8) then we precisely recover the Stokes equations derived in [45, 290], which can be used to obtain the DC conductivity, when finite, after setting the sources in the Stokes equations to zero. In fact since these DC Stokes equations with sources will be used later, we record them in appendix 3.C for reference.

Also notice that the system of equations does not form a closed set of equations for the perturbation when $\omega \neq 0$. In order to obtain a full solution, we also need to use the radial equations of motion. Interestingly, however, we will show in the next section that the constraint equations are sufficient to extract the dispersion relation for the quasinormal diffusion modes. In appendix 3.D we discuss how the data provided at the horizon and at the AdS boundary allows one, in principle, to solve the full set of Einstein equations.

3.4 Constructing the bulk diffusion perturbations

In this section we explain how one can systematically construct quasi-normal modes that are associated with diffusion of heat and charge. We construct these source-free modes in a long wavelength 'hydrodynamic expansion' that is valid for an arbitrary background black hole solution (3.2.2). While the explicit construction of these modes require that one solves both the constraint equations at the horizon as well radial equations in the bulk, it is possible to show that the leading order dispersion relation for the diffusion modes can be expressed in terms of the static susceptibilities as well as the 'horizon DC conductivity' obtained from a Stokes flow on the horizon given in appendix 3.C. Now for holographic lattices, when translation invariance is broken explicitly, the horizon DC conductivity is the same as the DC conductivity of the dual field theory. Thus, our result corresponds to a derivation of an Einstein relation.

We begin by describing a zero mode perturbation that is constructed from thermodynamic considerations. We start with the background ansatz (3.2.2) and then vary the temperature T and the zero mode of the chemical potential $\bar{\mu}$ via $T+\delta T$ and $\bar{\mu}+\delta\bar{\mu}$, where δT,$\delta\bar{\mu}$ are real constants. This gives rise to a 'thermodynamic perturbation' of the metric, gauge field and scalar field of the form $\delta g^{TH}_{\mu\nu} = \frac{\partial g_{\mu\nu}}{\partial T}\,\delta T + \frac{\partial g_{\mu\nu}}{\partial\bar{\mu}}\,\delta\bar{\mu}$, $\delta A^{TH} = (\frac{\partial a_t}{\partial T}\,\delta T + \frac{\partial a_t}{\partial\bar{\mu}}\,\delta\bar{\mu})\,dt$ and $\delta\phi^{TH} = \frac{\partial\phi}{\partial T}\,\delta T + \frac{\partial\phi}{\partial\bar{\mu}}\,\delta\bar{\mu}$, respectively. By considering the asymptotic behaviour of the background black holes, given in (3.2.3), we see that this perturbation has no source terms for the metric and the scalar, but there is a new source term for the gauge-field of the form $\delta\bar{\mu}$. As we are interested in source free solutions, we will deal with this in a moment.

We next observe that close to the horizon the above perturbation approaches

$$\begin{aligned}
\delta ds^2 &= -\frac{\delta T}{T}\left(4\pi T r\,dt^2 + \frac{dr^2}{4\pi T r}\right) + \left(\frac{\partial g^{(0)}_{ij}}{\partial T}\,\delta T + \frac{\partial g^{(0)}_{ij}}{\partial\bar{\mu}}\,\delta\bar{\mu}\right) dx^i dx^j + \cdots\,,\\
\delta a^{TH}_t &= r\left(\frac{\partial a^{(0)}_t}{\partial T}\,\delta T + \frac{\partial a^{(0)}_t}{\partial\bar{\mu}}\,\delta\bar{\mu}\right) + \cdots\,,\\
\delta\phi^{TH} &= \frac{\partial\phi^{(0)}}{\partial T}\,\delta T + \frac{\partial\phi^{(0)}}{\partial\bar{\mu}}\,\delta\bar{\mu}\,.
\end{aligned} \tag{3.4.1}$$

Notice that this does not satisfy the regularity conditions (3.3.2)-(3.3.4) required of a real time perturbation. To remedy this, and also to remove the extra source term in the gauge field, we perform a time coordinate transformation $t \to t + \frac{\delta T}{T}\,g(r)$ with $g(r)$ vanishing sufficiently fast as $r \to \infty$ and $g(r) = \ln r/(4\pi T) + g^{(1)} r + \ldots$ as $r \to 0$. We also perform the gauge transformation $\delta A^{RT} = \delta A^{TH} + d\Lambda$ with $\Lambda = -(t + g(r))\delta\bar{\mu}$. After performing these transformations we will denote the perturbation with a superscript RT for 'real-time'.

At the horizon this RT perturbation approaches

$$\begin{aligned}
\delta ds^2 = \delta g^{RT}_{\mu\nu}\,dx^\mu dx^\nu = -\frac{\delta T}{T}\left(4\pi T r\,dt^2 + \frac{dr^2}{4\pi T r}\right) - 2\,\frac{\delta T}{T}\,dt\,dr\\
+ \left(\frac{\partial g^{(0)}_{ij}}{\partial T}\,\delta T + \frac{\partial g^{(0)}_{ij}}{\partial\bar{\mu}}\,\delta\bar{\mu}\right) dx^i dx^j + \cdots\,,
\end{aligned}$$

$$\begin{aligned}
\delta a_t^{RT} &= -\delta\bar{\mu} + r\left(\frac{\partial a_t^{(0)}}{\partial T}\,\delta T + \frac{\partial a_t^{(0)}}{\partial \bar{\mu}}\,\delta\bar{\mu}\right) + \cdots\,,\\
\delta a_r^{RT} &= -\delta\bar{\mu}\,(4\pi T\,r)^{-1} + \frac{\delta T}{T}(4\pi T)^{-1} a_t^{(0)} - g^{(1)}\delta\bar{\mu} + \cdots\,,\\
\delta\phi^{RT} &= \frac{\partial \phi^{(0)}}{\partial T}\,\delta T + \frac{\partial \phi^{(0)}}{\partial \bar{\mu}}\,\delta\bar{\mu} + \cdots\,.
\end{aligned} \tag{3.4.2}$$

From this we can read off the near horizon quantities using the notation of equation (3.3.2)-(3.3.4)

$$\begin{aligned}
&\delta g_{tt}^{RT(0)} = \delta g_{rr}^{RT(0)} = \delta g_{tr}^{RT(0)} = -\frac{\delta T}{T},\\
&\delta g_{ti}^{RT(0)} = 0, \quad \delta g_{ri}^{RT(0)} = 0\,, \quad \delta g_{ij}^{RT(0)} = \frac{\partial g_{ij}^{(0)}}{\partial T}\,\delta T + \frac{\partial g_{ij}^{(0)}}{\partial \bar{\mu}}\,\delta\bar{\mu}\,,\\
&\delta a_t^{RT(0)} = -\delta\bar{\mu}, \quad \delta a_t^{RT(1)} = \frac{\partial a_t^{(0)}}{\partial T}\,\delta T + \frac{\partial a_t^{(0)}}{\partial \bar{\mu}}\,\delta\bar{\mu}\,,\\
&\delta a_r^{RT(0)} = -\delta\bar{\mu}, \quad \delta a_r^{RT(1)} = \frac{\delta T}{T}\,a_t^{(0)} - 4\pi T g^{(1)}\delta\bar{\mu}\,,\\
&\delta a_i^{RT(0)} = 0\,, \qquad \delta\phi^{RT(0)} = \frac{\partial \phi^{(0)}}{\partial T}\,\delta T + \frac{\partial \phi^{(0)}}{\partial \bar{\mu}}\,\delta\bar{\mu}\,.
\end{aligned} \tag{3.4.3}$$

Notice that this perturbation has $v^i = 0$, $w = -\delta\bar{\mu}$ and $p = 4\pi\delta T$ which clearly solves (3.3.6)-(3.3.8) for vanishing frequency, $\omega = 0$.

We now introduce a small parameter ε which will be used to perturbatively construct a real time diffusive mode. Following [1], and recalling (3.3.1), the perturbation is taken to be of the form

$$\begin{aligned}
\delta g_{\mu\nu} &= e^{-i\omega[t+S(r)]+i\varepsilon k_i x^i}\left(\delta g_{\mu\nu}^{RT} + \varepsilon\,\delta g_{[1]\mu\nu} + \varepsilon^2\,\delta g_{[2]\mu\nu} + \cdots\right)\,,\\
\delta A_\mu &= e^{-i\omega[t+S(r)]+i\varepsilon k_i x^i}\left(\delta A_\mu^{RT} + \varepsilon\,\delta A_{[1]\mu} + \varepsilon^2\,\delta A_{[2]\mu} + \cdots\right)\,,\\
\delta\phi &= e^{-i\omega[t+S(r)]+i\varepsilon k_i x^i}\left(\delta\phi^{RT} + \varepsilon\,\delta\phi_{[1]} + \varepsilon^2\,\delta\phi_{[2]} + \cdots\right)\,,
\end{aligned} \tag{3.4.4}$$

with the corrections $\delta g_{[m]\mu\nu}$, $\delta A_{[m]\mu}$, $\delta\phi_{[m]}$, $m = 1, 2, \ldots$ being time independent, complex functions of (r, x^i) that are periodic in the spatial coordinates x^i. We demand that order by order in the expansion in ε, the corrections have near horizon expansions analogous to (3.3.2)-(3.3.4). Specifically,

$$\begin{aligned}
&\delta g_{[m]tt} = (4\pi T r)\left(\delta g_{[m]tt}^{(0)}(x) + \mathcal{O}(r)\right)\,, \qquad \delta g_{[m]rr} = \frac{1}{(4\pi T r)}\left(\delta g_{[m]rr}^{(0)}(x) + \mathcal{O}(r)\right),\\
&\delta g_{[m]ij} = \delta g_{[m]ij}^{(0)}(x) + \mathcal{O}(r), \qquad \delta g_{[m]tr} = \delta g_{[m]tr}^{(0)}(x) + \mathcal{O}(r)\,,\\
&\delta g_{[m]tj} = \delta g_{[m]tj}^{(0)}(x) + r\delta g_{[m]tj}^{(1)}(x) + \ldots,\\
&\delta g_{[m]rj} = \frac{1}{(4\pi T r)}\left(\delta g_{[m]rj}^{(0)}(x) + r\delta g_{[m]rj}^{(1)}(x) + \ldots\right),
\end{aligned}$$

$$
\begin{aligned}
&\delta A_{[m]t} = \delta a^{(0)}_{[m]t}(x) + r\,\delta a^{(1)}_{[m]t}(x) + \cdots\,,\\
&\delta A_{[m]r} = \frac{1}{(4\pi T r)}\left(\delta a^{(0)}_{[m]r}(x) + r\,\delta a^{(1)}_{[m]r}(x) + \cdots\right)\,,\\
&\delta A_{[m]j} = \delta a^{(0)}_{[m]j}(x) + \mathcal{O}(r)\,,\quad \delta\phi_{[m]} = \delta\phi^{(0)}_{[m]}(x) + \mathcal{O}(r)\,,
\end{aligned}
\tag{3.4.5}
$$

with the analogue of the conditions in (3.3.4) satisfied for each $[m]$.

3.4.1 Dispersion relations for the diffusion modes

We now explain how we can obtain the dispersion relations for the perturbation, order by order as an expansion in ε. We will first show how solving the constraint equations (3.3.6)-(3.3.8) on the horizon, to a certain order in ε, is enough to obtain the leading order dispersion relation for ω as a function of the wave vector k_i in terms of the horizon DC conductivity, obtained from a Stokes flow on the horizon, and the thermodynamic susceptibilities. As already noted, the arguments in appendix 3.D ensure that the full perturbation will solve all the equations of motion. Some additional subtleties are discussed in appendix 3.E.

From (3.4.4),(3.4.5) and the analogue of (3.3.4), the expansion at the horizon that we consider takes the form

$$
\begin{aligned}
\omega &= \varepsilon\,\omega_{[1]} + \varepsilon^2\,\omega_{[2]} + \cdots\,,\\
p &= e^{i\varepsilon k_i x^i}\left(4\pi\delta T + \varepsilon\,p_{[1]} + \varepsilon^2\,p_{[2]} + \cdots\right)\,,\\
v_i &= e^{i\varepsilon k_i x^i}\left(\varepsilon\,v_{[1]i} + \varepsilon^2\,v_{[2]i}\cdots\right)\,,\\
w &= e^{i\varepsilon k_i x^i}\left(-\delta\bar\mu + \varepsilon\,w_{[1]} + \varepsilon^2\,w_{[2]} + \cdots\right)\,,\\
\delta g^{(0)}_{ij} &= e^{i\varepsilon k_i x^i}\left(\frac{\partial g^{(0)}_{ij}}{\partial T}\,\delta T + \frac{\partial g^{(0)}_{ij}}{\partial\bar\mu}\,\delta\bar\mu + \varepsilon\,\delta g^{(0)}_{[1]ij} + \cdots\right)\,,\\
\delta\phi^{(0)} &= e^{i\varepsilon k_i x^i}\left(\frac{\partial\phi^{(0)}}{\partial T}\,\delta T + \frac{\partial\phi^{(0)}}{\partial\bar\mu}\,\delta\bar\mu + \varepsilon\,\delta\phi^{(0)}_{[1]} + \cdots\right)\,,
\end{aligned}
\tag{3.4.6}
$$

where $v_{[m]i} \equiv -\delta g^{(0)}_{[m]ti}$, $w_{[m]} \equiv \delta a^{(0)}_{[m]t}$ and $p_{[m]} \equiv -4\pi T\delta g^{(0)}_{[m]rt}$ (see (3.4.5)). At leading order in ε, the scalar constraint equations (3.3.6) and (3.3.7) read

$$
\begin{aligned}
&\nabla_i v^i_{[1]} = \frac{i\omega_{[1]}}{2}\left(\delta T\,g^{ij}_{(0)}\frac{\partial g^{(0)}_{ij}}{\partial T} + \delta\bar\mu\,g^{ij}_{(0)}\frac{\partial g^{(0)}_{ij}}{\partial\bar\mu}\right)\,,\\
&\nabla_j\left(Z^{(0)}\left(-i\,k^j\delta\bar\mu + \nabla^j w_{[1]} + v^j_{[1]}a^{(0)}_t\right)\right) =\\
&\qquad i\omega_{[1]}\left(\frac{1}{2}Z^{(0)}a^{(0)}_t g^{ij}_{(0)}\frac{\partial g^{(0)}_{ij}}{\partial T} + \partial_\phi Z^{(0)}a^{(0)}_t\,\frac{\partial\phi^{(0)}}{\partial T} + Z^{(0)}\frac{\partial a^{(0)}_t}{\partial T}\right)\delta T\\
&\qquad + i\omega_{[1]}\left(\frac{1}{2}Z^{(0)}a^{(0)}_t g^{ij}_{(0)}\frac{\partial g^{(0)}_{ij}}{\partial\bar\mu} + \partial_\phi Z^{(0)}a^{(0)}_t\,\frac{\partial\phi^{(0)}}{\partial\bar\mu} + Z^{(0)}\frac{\partial a^{(0)}_t}{\partial\bar\mu}\right)\delta\bar\mu\,.
\end{aligned}
\tag{3.4.7}
$$

while the vector constraint equation (3.3.8) has the form

$$\begin{aligned}&-2\,\nabla^j\nabla_{(j}v_{[1]i)}-Z^{(0)}a_t^{(0)}\,\nabla_i w_{[1]}+\nabla_i\phi^{(0)}v_{[1]}^j\nabla_j\phi^{(0)}+ik_i4\pi\,\delta T+Z^{(0)}a_t^{(0)}\,ik_i\delta\bar{\mu}\\&+\nabla_i p_{[1]}-i\,\omega_{[1]}\,\nabla_i\phi^{(0)}\left(\frac{\partial\phi^{(0)}}{\partial T}\,\delta T+\frac{\partial\phi^{(0)}}{\partial\bar{\mu}}\,\delta\bar{\mu}\right)\\&+i\,\omega_{[1]}\,\nabla^k\left(\frac{\partial g_{ki}^{(0)}}{\partial T}\,\delta T+\frac{\partial g_{ki}^{(0)}}{\partial\bar{\mu}}\,\delta\bar{\mu}\right)=0\end{aligned}\tag{3.4.8}$$

At this point we pause to comment on the structure of this system of equations, which will have echoes at higher orders. Specifically, they are a sourced version of the horizon Stokes flow equations which were identified in [45, 290] to calculate the DC conductivity (see (3.C.1),(3.C.2)). In particular, the temperature gradient and electric field in (3.C.1),(3.C.2) are given by $\bar{\zeta}_i=-ik_i\delta T/T$ and $\bar{E}_i=-ik_i\delta\bar{\mu}$ and the source terms are parametrised by $\omega_{[1]}$. Following the arguments of [45, 290], as long as the horizon does not have any Killing vectors, the unknown variables $w_{[1]}$, $p_{[1]}$ and $v_{[1]}^i$ are fixed up to global shifts of the horizon scalars $w_{[1]}$ and $p_{[1]}$ by constants which we call $\delta\bar{\mu}_{[1]}$ and $4\pi\,\delta T_{[1]}$. We therefore see that it is not possible at this order in perturbation theory to fix these horizon zero modes for the functions $w_{[1]}$ and $p_{[1]}$. However, imposing periodic boundary conditions puts strong constraints on the sources of these equations which appear on the right hand side. On one hand, it will be one of the significant ingredients in fixing the frequency ω order by order. On the other hand, as we will see in appendix 3.E, the constants $\delta\bar{\mu}_{[1]}$ and $4\pi\,\delta T_{[1]}$ will be fixed by demanding existence of $w_{[3]}$ and $p_{[3]}$ i.e. at third order in perturbation theory. This is the structure one encounters at each order in the ε expansion. For bookkeeping, we will subtract the zero modes according to

$$w_{[i]}=\hat{w}_{[i]}+\delta\bar{\mu}_{[i]},\qquad p_{[i]}=\hat{p}_{[i]}+4\pi\,\delta T_{[i]}\tag{3.4.9}$$

with the hatted variables having zero average over a period and are therefore uniquely fixed after solving the system of constraints.

To proceed, we multiply by $\sqrt{g_{(0)}}$, and integrate the above equations over a spatial period. Using the definitions of the thermodynamic susceptibilities given in (3.2.10) we obtain two conditions which can be written in matrix form as

$$i\omega_{[1]}\begin{pmatrix}T^{-1}c_\mu & \xi\\ \xi & \chi\end{pmatrix}\begin{pmatrix}\delta T\\ \delta\bar{\mu}\end{pmatrix}=0\,.\tag{3.4.10}$$

Assuming that the matrix of susceptibilities is an invertible matrix, which is generically the case, we deduce that $\omega_{[1]}=0$. For later reference, notice that we do not assume here that the matrix is positive definite as required for thermodynamical stability. Thus, at this order in ε the full set of constraints (3.3.6)-(3.3.8) reduce to

$$\begin{aligned}
&\nabla_i v^i_{[1]} = 0\,,\\
&\nabla_j\left(Z^{(0)}\left(-i\,k^j\delta\bar\mu + \nabla^j w_{[1]} + v^j_{[1]}a_t^{(0)}\right)\right) = 0\,,\\
&-2\,\nabla^j\nabla_{(j}v_{[1]i)} + \nabla_i p_{[1]} - Z^{(0)}a_t^{(0)}\,\nabla_i w_{[1]} + v^j_{[1]}\nabla_j\phi^{(0)}\nabla_i\phi^{(0)}\\
&+ ik_i 4\pi\,\delta T + Z^{(0)}a_t^{(0)}\,ik_i\delta\bar\mu = 0\,,
\end{aligned} \tag{3.4.11}$$

which are now precisely the Stokes equations that are used in determining the DC conductivity given in (3.C.1),(3.C.2) provided that, as above, we identify the sources $\bar\zeta_i = -ik_i\delta T/T$ and $\bar E_i = -ik_i\delta\bar\mu$. These can be uniquely solved for $\hat w_{[1]}$, $\hat p_{[1]}$ and $v_{[1]}$ (provided that the horizon does not have Killing vectors). Thus, after integrating over the horizon we can deduce, in particular, that

$$\begin{aligned}
4\pi T i\oint\sqrt{g_{(0)}}\,v^i_{[1]} &= T\bar\alpha_H^{ij}k_j\delta\bar\mu + \bar\kappa_H^{ij}k_j\delta T\,,\\
i\oint\sqrt{g_{(0)}}Z^{(0)}\left(-i\,k^j\delta\bar\mu + \nabla^j w_{[1]} + v^j_{[1]}a_t^{(0)}\right) &= \sigma_H^{ij}k_j\delta\bar\mu + \alpha_H^{ij}k_j\delta T\,,
\end{aligned} \tag{3.4.12}$$

where σ_H^{ij}, α_H^{ij}, $\bar\alpha_H^{ij}$, $\bar\kappa_H^{ij}$ are the sub-matrices of the full thermoelectric horizon DC conductivity (see (3.C.4)). When the DC conductivities of the dual field theory are finite, as in the case of explicit breaking of translations, then these are in fact the same as the DC conductivity of the dual field theory, which we will denote by σ^{ij}, α^{ij}, $\bar\alpha^{ij}$, $\bar\kappa^{ij}$, respectively. For the time-reversal invariant backgrounds we are considering it will be useful to recall that σ and $\bar\kappa$ are symmetric matrices, while $\alpha = \bar\alpha^T$.

We now examine the constraint equations at second order in ε. The scalar constraints (3.3.6) and (3.3.7) give

$$\begin{aligned}
&\nabla_i v^i_{[2]} = \frac{i\omega_{[2]}}{2}\left(\delta T\,g^{ij}_{(0)}\frac{\partial g^{(0)}_{ij}}{\partial T} + \delta\bar\mu\,g^{ij}_{(0)}\frac{\partial g^{(0)}_{ij}}{\partial\bar\mu}\right) - ik_i v^i_{[1]}\,,\\
&\nabla_j\left(Z^{(0)}\left(i\,k^j\delta\bar\mu_{[1]} + \nabla^j\hat w_{[2]} + g^{ji}_{(0)}v_{[2]j}a_t^{(0)}\right)\right) =\\
&-ik_j\left(Z^{(0)}\left(-i\,k^j\delta\bar\mu + \nabla^j\hat w_{[1]} + g^{ji}_{(0)}v_{[1]j}a_t^{(0)}\right)\right) - i\nabla_j\left(Z^{(0)}k^j\hat w_{[1]}\right)\\
&+ i\omega_{[2]}\left(\frac{1}{2}Z^{(0)}a_t^{(0)}g^{ij}_{(0)}\frac{\partial g^{(0)}_{ij}}{\partial T} + \partial_\phi Z^{(0)}a_t^{(0)}\,\frac{\partial\phi^{(0)}}{\partial T} + Z^{(0)}\frac{\partial a_t^{(0)}}{\partial T}\right)\delta T\\
&+ i\omega_{[2]}\left(\frac{1}{2}Z^{(0)}a_t^{(0)}g^{ij}_{(0)}\frac{\partial g^{(0)}_{ij}}{\partial\bar\mu} + \partial_\phi Z^{(0)}a_t^{(0)}\,\frac{\partial\phi^{(0)}}{\partial\bar\mu} + Z^{(0)}\frac{\partial a_t^{(0)}}{\partial\bar\mu}\right)\delta\bar\mu\,.
\end{aligned} \tag{3.4.13}$$

and we will not explicitly write down the vector equation (3.3.8). We remind the reader that the constants $\delta\bar\mu_{[1]}$ and $\delta T_{[1]}$ have not been fixed yet and will be fixed by demanding existence of the solution at third order in the perturbative expansion, as discussed further in appendix 3.E. Furthermore, the zero modes $\delta\bar\mu_{[2]}$ and $4\pi\,\delta T_{[2]}$ of the functions $w_{[2]}$ and $p_{[2]}$ that we extracted according to (3.4.9) will only be fixed by demanding existence of the solution at fourth order in the ε expansion.

Even without knowing the value of the constants $\delta\bar{\mu}_{[1]}$ and $\delta T_{[1]}$, existence of the solution at second order will constrain the frequency $\omega_{[2]}$ and the constants $\delta\bar{\mu}$ and δT. Indeed, integrating the two equations (3.4.13) over a period we find that

$$\begin{aligned} i\omega_{[2]}\left(T^{-1}c_\mu\delta T+\xi\,\delta\bar{\mu}\right)-T^{-1}\bar{\kappa}^{ij}k_ik_j\,\delta T-\alpha^{ij}k_ik_j\,\delta\bar{\mu}=&0\,,\\ i\omega_{[2]}\left(\xi\,\delta T+\chi\,\delta\bar{\mu}\right)-\alpha^{ij}k_ik_j\,\delta T-\sigma^{ij}k_ik_j\,\delta\bar{\mu}=&0\,, \end{aligned} \tag{3.4.14}$$

where we used the fact that $\alpha^{ij}k_ik_j=\bar{\alpha}^{ij}k_ik_j$, for the backgrounds we are considering.

The algebraic system (3.4.14) is exactly the same as that considered in [1] (see also [231]) and we can immediately obtain the two eigenfrequencies $i\omega^{\pm}_{[2]}$ associated with the diffusion modes. Defining

$$\bar{\kappa}(k)\equiv\bar{\kappa}^{ij}k_ik_j,\quad \alpha(k)\equiv\alpha^{ij}k_ik_j,\quad \sigma(k)\equiv\sigma^{ij}k_ik_j\,, \tag{3.4.15}$$

we obtain the generalised Einstein relation

$$\begin{aligned} i\omega^{+}_{[2]}\,i\omega^{-}_{[2]}=&\frac{\kappa(k)}{c_\rho}\frac{\sigma(k)}{\chi}\,,\\ i\omega^{+}_{[2]}+i\omega^{-}_{[2]}=&\frac{\kappa(k)}{c_\rho}+\frac{\sigma(k)}{\chi}+\frac{T\,\left[\chi\,\alpha(k)-\xi\,\sigma(k)\right]^2}{c_\rho\chi^2\sigma(k)}\,, \end{aligned} \tag{3.4.16}$$

where $c_\rho=c_\mu-\frac{T\xi^2}{\chi}$ was given in (3.2.11) and we have defined

$$\kappa(k)\equiv\bar{\kappa}(k)-\frac{\alpha^2(k)T}{\sigma(k)}\,. \tag{3.4.17}$$

This is the universal result concerning the dispersion relations for the diffusive modes associated with the conserved heat and electric currents for holographic lattices.

3.4.2 Comments

Recall that $\bar{\kappa}^{ij}$ is the thermal DC conductivity for zero applied electric field. On the other hand the thermal DC conductivity for zero electric current, κ^{ij}, is given by $\kappa^{ij}=\bar{\kappa}^{ij}-T(\bar{\alpha}\sigma^{-1}\alpha)^{ij}$. Despite the notation, note that, in general, $\kappa(k)\neq\kappa^{ij}k_ik_j$.

The simplest dispersion relations occur for charge neutral background black holes with vanishing gauge fields. In this case we have $\alpha^{ij}=\xi=0$ and hence $\bar{\kappa}^{ij}=\kappa^{ij}$

leading to the simple Einstein relations[7]

$$i\omega = \varepsilon^2 \frac{\kappa^{ij} k_i k_j}{c_\mu} + \dots\,, \qquad i\omega = \varepsilon^2 \frac{\sigma^{ij} k_i k_j}{\chi} + \dots\,. \tag{3.4.18}$$

In the special case of translationally invariant black holes with vanishing gauge fields[8], the electric DC conductivity of the dual field theory is still finite and there is a corresponding charge diffusive mode as in (3.4.18). On the other hand the thermal DC conductivity is infinite and there is no thermal diffusive mode.

To make some additional comments, we first recall some results concerning the DC conductivity for perturbative holographic lattices for which translations are broken weakly. Such lattices have a black hole horizon that is a perturbation about a flat horizon, parametrised by a small number λ. In [45, 290] it was shown that the DC conductivity has the expansion

$$\begin{aligned}
&\bar{\kappa}^{ij} = (L^{-1})^{ij} 4\pi s T + \mathcal{O}(\lambda^{-1})\,, \; \alpha^{ij} = \bar{\alpha}^{ij} = (L^{-1})^{ij} 4\pi\rho + \mathcal{O}(\lambda^{-1})\,,\\
&\sigma^{ij} = (L^{-1})^{ij} \frac{4\pi\rho^2}{s} + \mathcal{O}(\lambda^{-1})\,.
\end{aligned} \tag{3.4.19}$$

where the matrix L_{ij} is proportional to λ^2 and depends on the leading order deviations of the horizon from the translationally invariant configuration. An explicit expression for L in terms of the spatially modulated horizon was given in [45, 290]. It was also shown in [45, 290] that both the thermal DC conductivity at zero current flow, κ^{ij}, and the electric conductivity at zero heat current flow, $\sigma^{ij}_{Q=0}$, appear at a higher order in the expansion. Explicitly, when $\rho \neq 0$ we have

$$\sigma^{ij}_{Q=0} = \frac{1}{4\pi} s Z^{(0)} g^{ij}_{(0)}\Big|_{\lambda=0} + \mathcal{O}(\lambda)\,, \qquad T\kappa^{ij} = \frac{s^3 T^2 Z^{(0)} g^{ij}_{(0)}}{4\pi\rho^2}\Bigg|_{\lambda=0} + \mathcal{O}(\lambda)\,. \tag{3.4.20}$$

Notice, in particular, $T\kappa^{ij} = \frac{s^2T^2}{\rho^2}\sigma^{ij}_{Q=0} + \mathcal{O}(\lambda)$. When $\rho = 0$ the expression for $\sigma^{ij}_{Q=0}$ is still valid but we can no longer calculate $T\kappa^{ij}$ perturbatively as the leading order piece is infinite.

Using the results (3.4.19) in (3.4.16) we obtain the dispersion relations of the two diffusive modes:

$$\begin{aligned}
i\omega^+ &= \frac{\rho^2\chi}{c_\rho \rho^2 \chi + T(\xi\rho - s\chi)^2} \kappa^{ij} k_i k_j + \mathcal{O}(\lambda)\,,\\
&= \frac{(sT)^2}{(sT)^2\chi - 2(sT\rho)T\xi + \rho^2 T c_\mu} \sigma^{ij}_{Q=0} k_i k_j + \mathcal{O}(\lambda)\,,
\end{aligned}$$

[7]In a charge neutral holographic setting, with translations explicitly broken using scalar fields as in [279], the first diffusive mode in (3.4.18) was numerically constructed in [238].

[8]This is the setting where the holographic Einstein relation for electric charge diffusion was first discussed in [85].

$$i\omega^- = \frac{4\pi(c_\rho \rho^2 \chi + T(\xi\rho - s\chi)^2)}{c_\rho s \chi^2}(L^{-1})^{ij} k_i k_j + \mathcal{O}(\lambda^{-1})\,. \tag{3.4.21}$$

In particular, the first diffusive mode is of order λ^0, but the second is of order λ^{-2} and hence appears parametrically further down the imaginary axis (while, of course, still going to the origin when $k_i \to 0$). If we now consider the translationally invariant case by taking $\lambda \to 0$, we find that we have one diffusive mode with a dispersion relation that satisfies an Einstein relation in terms of the finite DC conductivity κ^{ij}, or $\sigma^{ij}_{Q=0}$ (which is also valid when $\rho = 0$), as given in the first two lines of (3.4.21). Such a diffusive mode was discussed in the context of hydrodynamics in [134].

We can also consider spontaneous breaking of translations with the addition of a small explicit breaking, with dimensionless strength λ, as recently discussed for specific helical lattices in [340]. For temperatures just below T_c one can develop a double expansion in both λ and $(1 - T/T_c)^{1/2}$, with the exponent in the latter the expected behaviour for standard mean field phase transitions. The matrix L_{ij} in (3.4.19) will then also have have a double expansion. For the case that $\lambda >> (1 - T/T_c)^{1/2}$ we can expand, schematically, $L^{-1} \sim \lambda^{-2}[a_1 + \dots]$, where a_1 is a horizon quantity that can be calculated as in [45, 290] and the neglected terms are a double expansion in $(1 - T/T_c)^{1/2}/\lambda$ and λ. In particular, in this limit we see that the DC conductivity is dominated by the explicit breaking terms, as expected. Similarly, for $\lambda << (1 - T/T_c)^{1/2}$ we have $L^{-1} \sim (1 - T/T_c)^{-1}[a_2 + \dots]$, where the neglected terms are an expansion in $\lambda/(1 - T/T_c)^{1/2}$ and $(1 - T/T_c)^{1/2}$. This result explains a feature of the DC conductivity that was found numerically in figure 9 of [340]. It is also worth noting that in the case that $\lambda << (1 - T/T_c)^{1/2}$ this drop in the DC conductivity, combined with sum rules, implies that in the AC conductivity the spectral weight will move from the Drude peak to mid frequencies, as seen in the example of [340]. For the case of spontaneous breaking with a small explicit breaking there will, of course, still be two diffusive modes with dispersion relations as in (3.4.21), and both can be expanded in terms of λ and $(1 - T/T_c)^{1/2}$. Note that in [340], for a specific setting of pinned helical phases, only the first diffusive mode in (3.4.21) was discussed.

Our final comment concerns instabilities of the background black hole solutions. In particular, the dispersion relations for the diffusive modes given in (3.4.16) allow us to make sharp statements concerning the relation between thermodynamic instability and dynamical instability of the holographic lattice black hole solutions. In the simplest case, when the gauge field is zero we know that when c_μ or χ is negative then we have a thermodynamic instability. But from (3.4.18) we immediately deduce that there is a quasinormal mode with a pole in the upper half of the complex frequency plane and this leads to a dynamical instability of the black hole solution.

Turning now to general black hole backgrounds with non-vanishing gauge-field, we can write the equation for the diffusive modes in (3.4.14) as

$$\left[i\omega_{[2]}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \chi & T\xi \\ T\xi & Tc_\mu \end{pmatrix}^{-1}\begin{pmatrix} \sigma(k) & T\alpha(k) \\ T\alpha(k) & T\bar{\kappa}(k) \end{pmatrix}\right]\begin{pmatrix} \delta\bar{\mu} \\ \delta T/T \end{pmatrix} = 0\,. \quad (3.4.22)$$

Since the thermoelectric horizon DC conductivity is a positive definite matrix, we see that a negative eigenvalue in the susceptibility matrix appearing in (3.4.22), associated with a thermodynamic instability, will again give rise to a quasi-normal mode in the upper half complex frequency plane and, correspondingly, a dynamical instability for the black hole.

3.5 Final comments

In this paper we showed how the quasi-normal modes associated with heat and charge diffusion can be constructed for holographic lattices in a long-wavelength, perturbative expansion. In particular the construction allowed us to derive the dispersion relation of the diffusive modes in terms of horizon DC conductivities, obtained from solutions to a Stokes flow on the horizon, and static susceptibilities. This constitutes a derivation of a generalised Einstein relation.

We considered a class of gravitational theories with a specific matter content, but it is clear that the main results should apply to more general theories, including the possibility of having more gauge fields in the bulk and hence additional conserved charges in the dual field theory. We focussed on studying static geometries for simplicity, but it should be possible to relax this condition utilising the holographic understanding of transport currents presented in [291, 293]. Similarly, the extension to higher derivative theories of gravity should also be possible using the results in [291].

The derivation of the dispersion relations started with the identification of the quasinormal mode at $\omega = k_i = 0$, namely (3.4.3). This was possible because this diffusion mode is associated with conserved quantities. This was then used to perturbatively construct the quasinormal modes in a neighbourhood of $\omega = k_i = 0$. In particular, the analysis of the constraint equations on the stretched horizon was sufficient to obtain the dispersion relation for the quasinormal mode. It is clear that this procedure will work for the quasinormal modes associated with any conserved quantity[9].

[9] It also seems likely that if one is given a specific quasinormal mode for some $(\omega_0, k_0) \neq 0$ then it should also be possible to obtain the dispersion relation for the mode for (ω, k) close to (ω_0, k_0). However, in the case, the technical effort required to obtain the specific quasinormal mode probably allows one to construct the quasinormal mode for $(\omega, k) \neq (\omega_0, k_0)$ and so it is not clear if this

There has been some interesting recent discussion of the Goldstone modes that arise from spontaneously broken translation invariance, with an emphasis on the pinning phenomenon that occurs after adding in a small explicit breaking of translations, both within holography [340–342] and from a hydrodynamic point of view [237, 301, 302]. In the future, we plan to report on how the methods developed in this paper can be extended to study these modes as well as the Goldstone modes arising in spontaneously broken internal symmetries.

3.A Residual gauge invariance

The time dependent perturbation that we introduced in section 3.3 satisfied ingoing boundary conditions at the horizon summarised in (3.3.2)-(3.3.4). It is illuminating to note that there are some residual gauge and coordinate transformations which near the horizon are given by

$$\begin{aligned} \delta\Lambda &= e^{-i\omega v_{EF}}\,\Big(\delta\lambda^{(0)}(x^i) + r\,\delta\lambda^{(1)}(x^i) +, \mathcal{O}(r^2)\Big)\,,\\ t &\to t + e^{-i\omega v_{EF}}\,\Big(\delta T^{(0)}(x^j) + \mathcal{O}(r)\Big)\,,\\ r &\to r + e^{-i\omega v_{EF}}\,\Big(r\delta R^{(0)}(x^j) + \mathcal{O}(r^2)\Big)\,,\\ x^i &\to x^i + e^{-i\omega v_{EF}}\,\Big(\delta\xi^{(0)i}(x^j) + r\,\delta\xi^{(1)i}(x^j) + \mathcal{O}(r^2)\Big)\,. \end{aligned} \tag{3.A.1}$$

These are consistent with (3.3.2), (3.3.3) and induce the following transformations

$$\begin{aligned} &\delta g^{(0)}_{tt} \to \delta g^{(0)}_{tt} + 2i\omega\delta T^{(0)} - \delta R^{(0)}, \quad \delta g^{(0)}_{rr} \to \delta g^{(0)}_{rr} + (1 - \frac{2i\omega}{4\pi T})\delta R^{(0)},\\ &\delta g^{(0)}_{ij} \to \delta g^{(0)}_{ij} + 2\,\nabla_{(i}\delta\xi^{(0)}_{j)}, \qquad\qquad \delta g^{(0)}_{tr} \to \delta g^{(0)}_{tr} + i\omega\delta T^{(0)} - \frac{i\omega}{4\pi T}\delta R^{(0)}\,,\\ &\delta g^{(0)}_{ti} \to \delta g^{(0)}_{ti} - i\omega\delta\xi^{(0)}_i\,, \quad \delta g^{(1)}_{ti} \to \delta g^{(1)}_{ti} - 4\pi T\partial_i\delta T^{(0)} - i\omega(g^{(1)}_{ij}\delta\xi^{(0)j} + \delta\xi^{(1)}_i)\,,\\ &\delta g^{(0)}_{ri} \to \delta g^{(0)}_{ri} - i\omega\delta\xi^{(0)}_i\,, \quad \delta g^{(1)}_{ri} \to \delta g^{(1)}_{ri} + \partial_i\delta R^{(0)} - i\omega(g^{(1)}_{ij}\delta\xi^{(0)j} + \delta\xi^{(1)}_i) + 4\pi T\delta\xi^{(1)}_i\,, \end{aligned} \tag{3.A.2}$$

as well as

$$\begin{aligned} &\delta a^{(0)}_t \to \delta a^{(0)}_t - i\omega\delta\lambda^{(0)},\\ &\delta a^{(1)}_t \to \delta a^{(1)}_t - i\omega\delta\lambda^{(1)} - i\omega a^{(0)}_t\delta T^{(0)} + a^{(0)}_t\delta R^{(0)} + (\partial_i a^{(0)}_t)\delta\xi^{(0)i},\\ &\delta a^{(0)}_r \to \delta a^{(0)}_r - i\omega\delta\lambda^{(0)}, \qquad \delta a^{(1)}_r \to \delta a^{(1)}_r + (4\pi T - i\omega)\delta\lambda^{(1)} - i\omega a^{(0)}_t\delta T^{(0)}\,,\\ &\delta a^{(0)}_i \to \delta a^{(0)}_i + \partial_i\delta\lambda^{(0)}, \qquad \delta\phi^{(0)} \to \delta\phi^{(0)} + (\partial_i\phi^{(0)})\delta\xi^{(0)i}\,. \end{aligned} \tag{3.A.3}$$

One can check that the constraint equations given in (3.3.6)-(3.3.8) are covariant with respect to these transformations.

observation is that significant.

Notice that the combination $\delta a_t^{(1)} - \frac{i\omega}{4\pi T}(\delta a_t^{(1)} - \delta a_r^{(1)})$, appearing in the horizon constraint equation (3.3.7), is invariant under the gauge transformations parametrised by $\delta\lambda^{(1)}$. Similarly, the combination $\delta g_{ti}^{(1)} - \frac{i\omega}{4\pi T}(\delta g_{ti}^{(1)} - \delta g_{ri}^{(1)})$, appearing in (3.3.8), is invariant under the gauge transformations parametrised by $\delta\xi_i^{(1)}$. If $i\omega \neq 4\pi T$ we can choose $\delta\lambda^{(1)}$ to set $\delta a_r^{(1)} = 0$ and $\delta\xi_i^{(1)}$ to set $\delta g_{ri}^{(1)} = 0$, but we have not found a need to do this, nor fix any of the above gauge invariances[10].

3.B Evaluating the constraints on the horizon

3.B.1 Constraints in the radial decomposition

We begin by briefly summarising the constraint equations that appear in a Hamiltonian decomposition of the equations of motion using a radial foliation, following the presentation in appendix A of [45]. We introduce the normal vector n^μ, satisfying $n^\mu n_\mu = 1$. The D-dimensional metric $g_{\mu\nu}$ induces a $(D-1)$-dimensional Lorentzian metric on the slices of constant r via $h_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu$. The lapse and shift vectors are given by $n_\mu = N(dr)_\mu$ and $N^\mu = h^\mu{}_\nu r^\mu$, respectively, where $r^\mu = (\partial_r)^\mu$. The gauge-field components are decomposed via $b_\mu = h_\mu{}^\nu A_\nu$, $\Phi = -Nn^\mu A_\mu$ and we define $f_{\mu\nu} = \partial_\mu b_\nu - \partial_\nu b_\mu$.

The momenta conjugate to $h_{\mu\nu}$, b_μ and ϕ are given by

$$\begin{aligned}
\pi^{\mu\nu} &= -\sqrt{-h}\,\left(K^{\mu\nu} - K\,h^{\mu\nu}\right)\,,\\
\pi^\mu &= \sqrt{-h} Z F^{\mu\rho} n_\rho\,,\\
\pi_\phi &= -\sqrt{-h} n^\nu \partial_\nu \phi\,,
\end{aligned} \tag{3.B.1}$$

respectively, where $K_{\mu\nu} = \frac{1}{2}\mathcal{L}_n h_{\mu\nu}$ is the extrinsic curvature. The Hamiltonian, momentum and Gauss law constraints can then be written in the form $H = H^\nu = C = 0$ where

$$\begin{aligned}
H =& -(-h)^{-1/2}\left(\pi_{\mu\nu}\pi^{\mu\nu} - \frac{1}{D-2}\pi^2\right) - \sqrt{-h}\,\left({}^{(D-1)}R - V\right)\\
&-\frac{1}{2}(-h)^{-1/2} Z^{-1} h_{\mu\nu}\,\pi^\mu\,\pi^\nu + \frac{1}{4}\sqrt{-h}\,Z f_{\mu\nu}\,f_{\rho\sigma}\,h^{\mu\rho}\,h^{\nu\sigma}\\
&-\frac{1}{2}(-h)^{-1/2}\pi_\phi^2 + \frac{1}{2}\sqrt{-h}h^{\rho\sigma}\partial_\rho\phi\partial_\sigma\phi\,,\\
H^\nu =& -2\sqrt{-h}\,D_\mu\left((-h)^{-1/2}\pi^{\mu\nu}\right) + h^{\nu\sigma} f_{\sigma\rho}\pi^\rho\\
&-h^{\nu\sigma} b_\sigma \sqrt{-h} D_\rho\left((-h)^{-1/2}\,\pi^\rho\right) + h^{\nu\sigma}\partial_\sigma\phi\pi_\phi\,,\\
C =& \sqrt{-h}\,D_\mu\left((-h)^{-1/2}\,\pi^\mu\right)\,,
\end{aligned} \tag{3.B.2}$$

[10]We comment that a brief discussion of performing AC and DC calculations in a radial gauge, for Q-lattice constructions, appear in section 3 of [278] and in footnote 10 of [222].

where $\pi = \pi^\mu{}_\mu$ and D_μ is the covariant derivative with respect to $h_{\mu\nu}$.

The full equations of motion are the above constraint equations combined with the radial equations of motion. The latter consist of expressions for $\mathcal{L}_r h_{\mu\nu}$, $\mathcal{L}_r b_\mu$ and $\mathcal{L}_r \phi$ as well as $\mathcal{L}_r \pi^{\mu\nu}$,$\mathcal{L}_r \pi^{\mu}$,$\mathcal{L}_r \pi_\phi$ and explicit expressions can be found in appendix A of [45]. This leads to equations that have second order radial derivatives for $h_{\mu\nu}$, b_μ and ϕ.

3.B.2 Evaluating constraints for the perturbation

Consider a general perturbation of the background black hole solution (3.2.2) of the form $\delta g_{\mu\nu}$, δa_μ, $\delta\phi$ with all quantities functions of (t, r, x^i). On the surfaces of constant r this gives rise to a perturbed normal vector with components given by

$$\begin{aligned} n^i &= -(U/F)^{1/2}\, g_d^{ij}\, \delta g_{rj}, \qquad n^t = G^{-1}F^{-1/2}U^{-1/2}\delta g_{tr}, \\ n^r &= (U/F)^{1/2}\left(1 - \frac{U}{2F}\,\delta g_{rr}\right) . \end{aligned} \tag{3.B.3}$$

Furthermore, the corresponding shift and lapse functions are given by

$$\begin{aligned} N^j &= g_d^{ij}\, \delta g_{ri}, \qquad N^t = -\frac{1}{GU}\,\delta g_{rt}\,, \\ N &= (F/U)^{1/2}\left(1 + \frac{1}{2}\frac{U}{F}\,\delta g_{rr}\right) . \end{aligned} \tag{3.B.4}$$

The components of the extrinsic curvature $K_{\mu\nu}$ take the form

$$\begin{aligned} K^t{}_t =& \frac{1}{2}\, G^{-1}F^{-1/2}U^{-1/2}\left(\partial_r\,(GU) - \frac{1}{2}\frac{U}{F}\,\partial_r\,(GU)\ \delta g_{rr}\right) \\ &+ \frac{1}{2}\, G^{-2}F^{-1/2}U^{-3/2}\,\partial_r\,(GU)\,\delta g_{tt} \\ &- \frac{1}{2}\, G^{-1}F^{-1/2}U^{-1/2}\left(\partial_r \delta g_{tt} + \partial_j (GU)\, N^j\right) + G^{-1}F^{-1/2}U^{-1/2}\,\partial_t\delta g_{tr}\,, \\ K^i{}_t =& -\frac{1}{2}\, GF^{-1/2}U^{3/2}\, g_d^{ij}\left(-\partial_r\left(\frac{1}{GU}\,\delta g_{tj}\right) + \partial_j\left(\frac{1}{GU}\,\delta g_{rt}\right) + \frac{1}{GU}\partial_t\,\delta g_{rj}\right)\,, \\ K^t{}_i =& \frac{1}{2}\,(U/F)^{1/2}\left(-\frac{1}{GU}\partial_r\delta g_{ti} + \frac{\delta g_{tj}}{GU}g_d^{kj}\,\partial_r g_{dik} + \partial_i\left(\frac{1}{GU}\,\delta g_{rt}\right) + \frac{1}{GU}\partial_t\delta g_{ri}\right)\,, \\ K^i{}_j =& \frac{1}{2}\,(U/F)^{1/2}\left(g_d^{ik}\partial_r g_{dkj} - \frac{U}{2F}g_d^{ik}\partial_r g_{dkj}\delta g_{rr} + g_d^{ik}\partial_r\delta g_{kj} - g_d^{il}g_d^{km}\partial_r g_{dkj}\delta g_{lm}\right) \\ &- \frac{1}{2}(U/F)^{1/2}\left(\nabla^i N_j + \nabla_j N^i\right) . \end{aligned} \tag{3.B.5}$$

We now turn to the specific perturbation discussed in section 3.3. We want to evaluate the constraints at the horizon by employing the expansions given in (3.3.2)-(3.3.4). Expanding the extrinsic curvature near the horizon we obtain

$$K^t{}_t \to e^{-i\omega v_{EF}}\frac{1}{2}\frac{(4\pi T)^{1/2}}{r^{1/2}}\left(e^{i\omega v_{EF}} - \frac{1}{2}\delta g^{(0)}_{rr} - \frac{i\omega}{4\pi T}\delta g^{(0)}_{rr}\right)\,,$$

$$
\begin{aligned}
K^i{}_t &\to e^{-i\omega v_{EF}} \frac{1}{2} \frac{(4\pi T)^{1/2}}{r^{1/2}} v^i \,, \\
K^t{}_i &\to e^{-i\omega v_{EF}} \frac{1}{2} \frac{1}{(4\pi T)^{1/2}} \frac{1}{r^{1/2}} \left(-\delta g^{(1)}_{ti} + \partial_i \delta g^{(0)}_{tr} - g^{(1)}_{il} v^l + \frac{i\,\omega}{4\pi T} (\delta g^{(1)}_{ti} - \delta g^{(1)}_{ri}) \right) \,, \\
K^i{}_j &\to e^{-i\omega v_{EF}} \frac{1}{2} \frac{1}{(4\pi T)^{1/2}} \frac{1}{r^{1/2}} \left(\nabla^i v_j + \nabla_j v^i - i\omega\, g^{ik}_{(0)} \delta g^{(0)}_{kj} \right) \,, \\
K &\to e^{-i\omega v_{EF}} \frac{1}{2} \frac{(4\pi T)^{1/2}}{r^{1/2}} \left(e^{i\omega v_{EF}} - \frac{1}{2} \delta g^{(0)}_{rr} + \frac{2}{4\pi T} \nabla_i v^i - \frac{i\omega}{4\pi T} \left(\delta g^{(0)}_{rr} + g^{ij}_{(0)} \delta g^{(0)}_{ij} \right) \right) \,.
\end{aligned}
\tag{3.B.6}
$$

In the above we have only kept background terms at leading order $\mathcal{O}(r^{-1/2})$ since these are the only ones that will contribute in our calculation. Furthermore, the covariant derivative is with respect to the metric $g^{(0)}_{ij}$, which is also used to raise and lower indices.

We next consider the following quantity which appears in the momentum constraint

$$
\begin{aligned}
W_\nu =& D_\mu \left((-h)^{-1/2} \pi^\mu{}_\nu \right) = -D_\mu K^\mu{}_\nu + D_\nu K \,, \\
=& -(-h)^{-1/2}\, \partial_\mu \left(\sqrt{-h}\, K^\mu{}_\nu \right) + \frac{1}{2}\, \partial_\nu h_{\kappa\lambda} K^{\kappa\lambda} + \partial_\nu K \,.
\end{aligned}
\tag{3.B.7}
$$

Expanding at the horizon we find the following individual components

$$
\begin{aligned}
W_t &\to e^{-i\omega v_{EF}} \frac{(4\pi T)^{1/2}}{r^{1/2}} \left(-\frac{1}{2} \nabla_i v^i + \frac{i\omega}{4} g^{ij}_{(0)} \delta g^{(0)}_{ij} - \frac{i\omega}{8\pi T} \left(2\, \nabla_i v^i - i\omega g^{ij}_{(0)} \delta g^{(0)}_{ij} \right) \right) \,, \\
W_i &\to \frac{1}{2} e^{-i\omega v_{EF}} \frac{1}{(4\pi T)^{1/2}} \frac{1}{r^{1/2}} \Big[-2\, \nabla^j \nabla_{(j} v_{i)} + \nabla_i p' \\
&\quad + i\omega \left(-\delta g^{(1)}_{ti} + \partial_i \delta g^{(0)}_{tr} - g^{(1)}_{il} v^l + \frac{i\,\omega}{4\pi T} (\delta g^{(1)}_{ti} - \delta g^{(1)}_{ri}) + \nabla^k \delta g^{(0)}_{ki} \right) \Big] \,,
\end{aligned}
\tag{3.B.8}
$$

with

$$
p' = -2\pi T (\delta g^{(0)}_{tt} + \delta g^{(0)}_{rr}) + 2\, \nabla_j v^j + i\omega \left(\delta g^{(0)}_{tt} - 2\delta g^{(0)}_{tr} - g^{ij}_{(0)} \delta g^{(0)}_{ij} \right) \,. \tag{3.B.9}
$$

Another quantity that enters the constraints is the momentum of the scalar field. At leading order in r we have

$$
\pi_\phi \to -\sqrt{g_{(0)}}\, e^{-i\omega v_{EF}} \left(v^i \partial_i \phi^{(0)} - i\omega \delta\phi^{(0)} \right) \,. \tag{3.B.10}
$$

We next turn to the gauge field. We find

$$
\begin{aligned}
F_{tr} &\to -a^{(0)}_t + e^{-i\omega v_{EF}} \left(-\delta a^{(1)}_t + \frac{i\omega}{4\pi T} \left(\delta a^{(1)}_t - \delta a^{(1)}_r \right) \right) \,, \\
F_{ir} &\to e^{-i\omega v_{EF}} \frac{1}{4\pi T r} \left(\partial_i w + i\omega \delta a^{(0)}_i \right) + e^{-i\omega v_{EF}} \frac{1}{4\pi T} \left(\partial_i \delta a^{(1)}_r - 4\pi T \delta a^{(1)}_i + i\omega \delta a^{(1)}_i \right) , \\
F_{ti} &\to -e^{-i\omega v_{EF}} \left(\partial_i w + i\omega \delta a^{(0)}_i \right) - r \left(\partial_i a^{(0)}_t + e^{-i\omega v_{EF}} \partial_i \delta a^{(1)}_t + i\omega\, e^{-i\omega v_{EF}} \delta a^{(1)}_i \right) \,,
\end{aligned}
$$

$$
\begin{aligned}
F_{ij} &\to 2e^{-i\omega v_{EF}}\, \partial_{[i}\delta a^{(0)}_{j]} \,, \\
F^{tr} &\to a^{(0)}_t + e^{-i\omega v_{EF}} \Big[\delta a^{(1)}_t - \frac{i\omega}{4\pi T}\left(\delta a^{(1)}_t - \delta a^{(1)}_r\right) \\
&\qquad\qquad + \left(\delta g^{(0)}_{tt} - \delta g^{(0)}_{rr}\right) a^{(0)}_t + \frac{1}{4\pi T} g^{ij}_{(0)} v_j \partial_i a^{(0)}_t \Big] \,, \\
F^{ir} &\to e^{-i\omega v_{EF}}\, g^{ij}_{(0)} \left(\partial_j w + i\omega \delta a^{(0)}_j + v_j a^{(0)}_t\right) \,.
\end{aligned} \tag{3.B.11}
$$

and thus the associated momentum has the expansion

$$
\begin{aligned}
\pi^i &\to \sqrt{g_{(0)}} Z^{(0)} e^{-i\omega v_{EF}}\, g^{ij}_{(0)} \left(\partial_j w + i\omega \delta a^{(0)}_j + v_j a^{(0)}_t\right) \,, \\
\pi^t &\to \sqrt{g_{(0)}} Z^{(0)} e^{-i\omega v_{EF}} \left(\delta a^{(1)}_t - \frac{i\omega}{4\pi T}\left(\delta a^{(1)}_t - \delta a^{(1)}_r\right) + \frac{1}{4\pi T} g^{ij}_{(0)} v_j \partial_i a^{(0)}_t\right) \\
&\quad + \sqrt{g_{(0)}} Z^{(0)} e^{-i\omega v_{EF}} a^{(0)}_t \left(e^{i\omega v_{EF}} + \frac{1}{2}\left(\delta g^{(0)}_{tt} - \delta g^{(0)}_{rr} + g^{ij}_{(0)} \delta g^{(0)}_{ij}\right)\right) \\
&\quad + \sqrt{g_{(0)}}\, \partial_\phi Z^{(0)} e^{-i\omega v_{EF}} a^{(0)}_t \, \delta\phi^{(0)} \,.
\end{aligned} \tag{3.B.12}
$$

We can now evaluate the constraints at the horizon. Expanding the Gauss law constraint $C = \partial_\mu \pi^\mu = 0$ gives

$$
\begin{aligned}
&\nabla_i \left(Z^{(0)} \left(\nabla^i w + i\omega g^{ij}_{(0)} \delta a^{(0)}_j + v^i a^{(0)}_t\right)\right) = i\omega Z^{(0)} \frac{1}{2} a^{(0)}_t \left(\delta g^{(0)}_{tt} - \delta g^{(0)}_{rr} + g^{ij}_{(0)} \delta g^{(0)}_{ij}\right) \\
&+ i\omega Z^{(0)} \left(\delta a^{(1)}_t - \frac{i\omega}{4\pi T}\left(\delta a^{(1)}_t - \delta a^{(1)}_r\right) + \frac{1}{4\pi T} v^i \partial_i a^{(0)}_t\right) + i\omega\, \partial_\phi Z^{(0)} a^{(0)}_t \, \delta\phi^{(0)} \,.
\end{aligned} \tag{3.B.13}
$$

To expand the momentum constraints at the horizon, $H_\nu = 0$, we first note that

$$
\begin{aligned}
f_{t\mu}\pi^\mu &= F_{ti}\pi^i \to 0 \,, \\
f_{i\mu}\pi^\mu &= F_{it}\pi^t + F_{ij}\pi^j \to \sqrt{g_{(0)}} Z^{(0)} e^{-i\omega v_{EF}} a^{(0)}_t \left(\partial_i w + i\omega \delta a^{(0)}_i\right) \,,
\end{aligned} \tag{3.B.14}
$$

For the t component, $H_t = 0$, we then find

$$
\left(2\,\nabla_i v^i - i\omega g^{ij}_{(0)} \delta g^{(0)}_{ij}\right)\left(1 + \frac{i\omega}{2\pi T}\right) = 0 \,. \tag{3.B.15}
$$

Similarly for the i component $H_i = 0$ we get

$$
\begin{aligned}
&-2\,\nabla^j \nabla_{(j} v_{i)} + i\omega \left(-\delta g^{(1)}_{ti} + \partial_i \delta g^{(0)}_{tr} - g^{(1)}_{il} v^l + \frac{i\,\omega}{4\pi T}(\delta g^{(1)}_{ti} - \delta g^{(1)}_{ri}) + \nabla^k \delta g^{(0)}_{ki}\right) \\
&+ \nabla_i p' - Z^{(0)} a^{(0)}_t \left(\nabla_i w + i\omega \delta a^{(0)}_i\right) + \nabla_i \phi^{(0)} \left(v^j \nabla_j \phi^{(0)} - i\omega \delta\phi^{(0)}\right) = 0 \,.
\end{aligned} \tag{3.B.16}
$$

Finally, we consider the Hamiltonian constraint $(-h)^{-1/2} H = 0$. The third and fifth terms in (3.B.2) vanish at linearised order. The second and the sixth term is of order $\mathcal{O}(r^0)$. We also compute

$$
f_{ri} \to -\frac{i\omega e^{-i\omega v_{EF}}}{4\pi T r}\, \delta a^{(0)}_i + \mathcal{O}(r^0) \,,
$$

$$
\begin{aligned}
f_{rt} &\to -\frac{i\omega e^{-i\omega v_{EF}}}{4\pi T r}\, w\, + \mathcal{O}(r^0)\,, \\
f_{ti} &\to F_{ti}\,, \\
f_{ij} &\to F_{ij}\,,
\end{aligned}
\tag{3.B.17}
$$

and so the fourth term is of order $\mathcal{O}(r^0)$ as well. Finally, the first term turns out to be of order $\mathcal{O}(r^{-1})$, leading to the constraint

$$
2\,\nabla_i v^i - i\omega g^{ij}_{(0)}\delta g^{(0)}_{ij} = 0\,, \tag{3.B.18}
$$

which is consistent with (3.B.15). Finally, after using (3.3.4) and (3.B.18) in (3.B.16), we find that the latter takes the form

$$
\begin{aligned}
&i\omega\left(-\delta g^{(1)}_{ti} - g^{(1)}_{il}v^l + \partial_i(\delta g^{(0)}_{tr} - \delta g^{(0)}_{rr}) + \frac{i\,\omega}{4\pi T}(\delta g^{(1)}_{ti} - \delta g^{(1)}_{ri}) + \nabla^k\delta g^{(0)}_{ki}\right) \\
&\quad - 2\,\nabla^j\nabla_{(j}v_{i)} + \nabla_i p - Z^{(0)}a^{(0)}_t\left(\nabla_i w + i\omega\delta a^{(0)}_i\right) + \nabla_i\phi^{(0)}\left(v^j\nabla_j\phi^{(0)} - i\omega\delta\phi^{(0)}\right) = 0\,,
\end{aligned}
\tag{3.B.19}
$$

with

$$
p = -2\pi T(\delta g^{(0)}_{tt} + \delta g^{(0)}_{rr})\,. \tag{3.B.20}
$$

Thus, in summary, equations (3.B.13),(3.B.18) and (3.B.19) are the constraint equations for the perturbations on the horizon.

3.C Calculating the DC conductivity

We briefly summarise the results of [45, 290] which allows us to obtain a horizon DC conductivity by solving a system of Stokes equations on the horizon. When the DC conductivity of the dual field theory is finite, as in the case of explicit breaking of translations, it is identical to the horizon DC conductivity.

By analysing a perturbation of the background black hole solutions (3.2.2) that, crucially, incorporate DC sources, it was shown that one is led to the following system of Stokes equations on the black hole horizon

$$
\begin{gathered}
\partial_i Q^i_{(0)} = 0\,, \qquad \partial_i J^i_{(0)} = 0\,, \\
-2\,\nabla^i\nabla_{(i}v_{j)} - Z^{(0)}a^{(0)}_t\nabla_j w + \nabla_j\phi^{(0)}\nabla_i\phi^{(0)}v^i + \nabla_j\,p = 4\pi T\,\bar{\zeta}_j + Z^{(0)}a^{(0)}_t\bar{E}_j\,,
\end{gathered}
\tag{3.C.1}
$$

where

$$
Q^i_{(0)} = 4\pi T\sqrt{g_{(0)}}v^j\,,
$$

$$J^i_{(0)} = \sqrt{g_{(0)}} g^{ij}_{(0)} Z^{(0)} \left(\partial_j w + a_t^{(0)} v_j + \bar{E}_j \right) . \tag{3.C.2}$$

Here, the vectors $\bar{E}_i$,$\bar{\zeta}_i$, which are taken to be constant, parametrise the DC electric source and thermal gradient of the dual field theory, respectively.

After solving these Stokes equations we obtain the local currents $Q^i_{(0)}$, $J^i_{(0)}$ on the horizon as functions of the DC sources $\bar{E}_i, \bar{\zeta}_i$. By defining the current flux densities at the horizon via

$$\bar{J}^i_{(0)} = \int J^i_{(0)} \,, \qquad \bar{Q}^i_{(0)} = \int Q^i_{(0)} \,, \tag{3.C.3}$$

we can then define the horizon thermoelectric DC conductivity matrix via

$$\begin{pmatrix} \bar{J}^i_{(0)} \\ \bar{Q}^i_{(0)} \end{pmatrix} = \begin{pmatrix} \sigma^{ij}_H & T\alpha^{ij}_H \\ T\bar{\alpha}^{ij}_H & T\bar{\kappa}^{ij}_H \end{pmatrix} \begin{pmatrix} \bar{E}_j \\ \bar{\zeta}_j \end{pmatrix} . \tag{3.C.4}$$

For the time reversal invariant backgrounds we are considering in this paper we have σ_H, $\bar{\kappa}_H$ are symmetric and $\alpha_H = \bar{\alpha}^T_H$.

Furthermore, as explained in [45, 290], the current flux densities at the horizon, defined by

$$\bar{J}^i_{(0)} = \int J^i_{(0)} \,, \qquad \bar{Q}^i_{(0)} = \int Q^i_{(0)} \,, \tag{3.C.5}$$

are identical to the current fluxes $\bar{J}^i$, $\bar{Q}^i$ of the dual field theory. Thus, for holographic lattices we have the DC conductivities of the dual field theory, σ, α, $\bar{\alpha}$ and $\bar{\kappa}$, are identical to the horizon conductivities σ_H, α_H, $\bar{\alpha}_H$ and $\bar{\kappa}_H$, respectively.

It is helpful for the analysis of this paper to recall from [45, 290] that as long as the horizon does not have any Killing vectors, there is a unique solution to the Stokes equations (3.C.1), up to undetermined constants in w and p, which do not inhibit one solving for the DC conductivity since they do not enter the expressions for the currents.

Finally, we emphasise that the horizon DC conductivity given in (3.C.4) should not be confused with another notion of horizon conductivity that arises from the constitutive relations for the auxiliary fluid on the horizon. For example, in the expression for the electric current on the horizon given in (3.C.2), one can call $\sqrt{g_{(0)}} g^{ij}_{(0)} Z^{(0)}$ a local electric conductivity[11] , but this is, in general quite distinct from σ^{ij}_H as defined in (3.C.4).

[11] To avoid confusion, we note that in [290] the expression $\sqrt{g_{(0)}} g^{ij}_{(0)} Z^{(0)}$ was denoted by σ^{ij}_H, a notation which we do not use here.

3.D Counting functions of integration

The quasinormal diffusion modes are solutions to the bulk equations of motion satisfying ingoing boundary conditions at the black hole horizon and have vanishing source terms at the AdS boundary. In the text we focused on the constraint equations given in section 3.3.1 as this was sufficient to extract the diffusion relation for the modes. Here we outline how, with the time dependence given in (3.3.1), we have specified enough data at the horizon and at the AdS boundary in order to obtain a solution to the full equations of motion.

We begin with fluctuations of the scalar field which satisfies a second order equation in the radial variable. At the AdS boundary, $r \to \infty$, we have two functions of integration, depending on the spatial coordinates x^i, associated with the source terms and expectation value of the scalar operator in the dual CFT. At the black hole horizon we demand that the perturbation is regular and this leaves us with a single function of integration $\delta\phi^{(0)}(x)$. By setting the source terms at the AdS boundary to zero we can develop a solution in the bulk using the remaining function at the AdS boundary and then match with the solution developed from the horizon using $\delta\phi^{(0)}(x)$ which leads, generically, to a unique solution everywhere.

We next turn to fluctuations of the gauge field. The radial component δa_r serves as a Lagrange multiplier and is data which, *a priori*, we are free to specify. This leaves $D-1$ functions δa_i, δa_t, each of which satisfies a second order differential equation in the radial variable, and there is also the Gauss constraint, $C = 0$ (see (3.B.2)), that we impose infinitesimally close to the horizon, given in (3.3.7). At the AdS boundary we set the $D-1$ functions of integration that are associated with possible source terms to zero, implying that we need to identify $(D-1)$ functions from the horizon expansion in order to solve the second order equations of motion, via a matching argument. With the ingoing boundary conditions (3.3.3),(3.3.4) at the horizon, we have the functions $\delta a_t^{(0)}$ and, when $\omega \neq 0$, $\delta a_i^{(0)}$, $\delta a_t^{(1)}$ all appearing in the constraint equation. If we pick $\delta a_t^{(0)}$ to be solved by the constraint equation then we are left with precisely $D-1$ functions $\delta a_i^{(0)}$ and $\delta a_t^{(1)}$ which are fixed by the matching. It is worth noting that in our procedure, for the leading term of the Lagrange multiplier we must set $\delta a_r^{(0)} = w$ (see (3.3.4)), which we are free to do. In addition, we note that $\delta a_r^{(1)}$, the sub-leading term of the Lagrange multiplier, also appears in (3.3.7) and can be chosen freely; in particular $\delta a_r^{(1)}$ does not affect the in-falling conditions we have specified in (3.3.3), (3.3.4).

Finally, we discuss the metric fluctuations, which run along similar lines to the gauge field. There are $D(D+1)/2$ metric functions $\delta g_{\mu\nu}$ out of which the D functions $\delta g_{r\mu}$ serve as Lagrange multipliers. The remaining $D(D-1)/2$ functions, δg_{tt}, δg_{ti} and δg_{ij}, each satisfy differential equations which are second order in the radial direction. There are also D constraint equations, the Hamiltonian and momentum constraints, $H = H^\nu = 0$ (see (3.B.2)), that we have chosen to impose on a surface infinitesimally close to the horizon. Importantly, however, the Hamiltonian constraint is redundant close to the horizon leaving $D-1$ independent constraint equations to satisfy near the horizon, given in (3.3.6) and (3.3.8). There are now two equivalent ways to proceed, which we discuss in turn.

The first way is to solve the second order radial equations for δg_{tt}, δg_{ti} and δg_{ij}. Now with the ingoing boundary conditions (3.3.2), (3.3.4) these equations are associated with $D(D-1)/2 + (D-2)$ functions of integration on the horizon, $\delta g_{tt}^{(0)}$, $\delta g_{ij}^{(0)}$, $\delta g_{ti}^{(1)}$, and v_i, all of which appear in the constraint equations when $\omega \neq 0$ and we are ignoring the functions associated with the Lagrange multipliers appearing at the horizon for the moment. Close to the AdS boundary, where we fix the source terms to zero, the second order equations give a set of $D(D-1)/2$ of normalisable modes that will be fixed along with the $D(D-1)/2$ functions $\delta g_{tt}^{(0)}$, $\delta g_{ij}^{(0)}$, $\delta g_{ti}^{(1)}$ upon matching in the bulk. The remaining $D-1$ functions on the horizon, v_i and p, are then used to solve the momentum constraints $H^\nu = 0$. The issue that remains open in this approach is the Hamiltonian constraint. One can show that in an expansion close to the horizon this is satisfied order by order in an analytic radial expansion provided that the remaining constraint equations and second order equations are satisfied, along with imposing the boundary conditions (3.3.2), (3.3.3) and (3.3.4). Away from the horizon, this is guaranteed by the fact that $\mathcal{L}_r H = 0$.

The second way is to solve the Hamiltonian constraint equation in the bulk instead of the second order equation for δg_{tt}. Indeed, one of the second order radial equations, for example the one for δg_{tt}, is implied by the Hamiltonian constraint. This is because the Hamiltonian constraint contains no derivatives of the momentum and hence only $\partial_r \delta g_{tt}$ appears, along with second order spatial derivatives with respect to x^i. Thus, instead of the $D(D-1)/2$ second order equations in the radial direction, we just need to solve $D(D-1)/2 - 1$ second order equations and one first order equation. Setting the source terms to zero in these equations at the AdS boundary, we conclude that we need to specify $D(D-1)/2 - 1 + 1$ free functions at the horizon after imposing the ingoing boundary conditions and solving the $D-1$ constraint equations (3.3.6),(3.3.8). If we again use the $D-1$ constraint equations to solve for v_i, p, this will leave precisely $D(D-1)/2$ functions $\delta g_{ij}^{(0)}$, $\delta g_{tt}^{(0)}$, $\delta g_{ti}^{(1)}$ to be fixed by the matching.

Concerning the Lagrange multipliers, we first note that in the above procedure $\delta g_{rt}^{(0)} \equiv -p/(4\pi T)$ will be fixed. Furthermore, we must set $\delta g_{ri}^{(0)} = -v_i$, $\delta g_{rr}^{(0)} = -p/(2\pi T) - \delta g_{tt}^{(0)}$ (see (3.3.4)). In addition, we also note that the sub-leading term $\delta g_{ri}^{(1)}$ appears in (3.3.8) and can be chosen freely as part of fixing the Lagrange multipliers. Notice that, similarly to the case of the radial component of the gauge field $\delta a_r^{(1)}$, this does not spoil the in-falling conditions we have specified in (3.3.2), (3.3.4).

3.E Fixing the zero modes of the ε expansion

We now examine the perturbation at third order in ε. It is useful to split all the bulk fields $\mathbf{\Phi}_{[i]}$ according to

$$\mathbf{\Phi}_{[i]} = \hat{\mathbf{\Phi}}_{[i]} + \frac{\partial \mathbf{\Phi}_b}{\partial T}\,\delta T_{[i]} + \frac{\partial \mathbf{\Phi}_b}{\partial \bar{\mu}}\,\delta \bar{\mu}_{[i]}\,. \tag{3.E.1}$$

Here the second and third terms are the derivatives of the background solution with respect to the temperature and the averaged chemical potential, in the gauge described below (3.4.1); in particular at the horizon the derivatives are explicitly given in eq. (3.4.3). The functions $\hat{\mathbf{\Phi}}_{[i]}$ solve the perturbative in ε radial equations of motion with boundary conditions on the horizon set by $\hat{w}_{[i]}$, $\hat{p}_{[i]}$ and $v^j_{[i]}$ which are obtained from the perturbatively expanded horizon constraint equations and we recall that $\hat{w}_{[i]}$, $\hat{p}_{[i]}$ do not have a zero mode (see (3.4.9)). We stress that this doesn't necessarily mean that the bulk functions $\hat{\mathbf{\Phi}}$ do not have a zero mode. However, when $\mathbf{\Phi}$ is the field δa_t, for example, our boundary conditions impose that $\widehat{(\delta a_t)}_{[i]}$ is equal to $\hat{w}_{[i]}$ on the horizon and that function does not have a zero mode.

In this notation, at third order, the scalar constraint equations (3.3.6), (3.3.7) read

$$\begin{aligned}
\nabla_i v^i_{[3]} = {}& \frac{i\omega_{[3]}}{2}\left(g^{ij}_{(0)}\frac{\partial g^{(0)}_{ij}}{\partial T}\,\delta T + g^{ij}_{(0)}\frac{\partial g^{(0)}_{ij}}{\partial \bar{\mu}}\,\delta\bar{\mu}\right) \\
&+ \frac{i\omega_{[2]}}{2}\left(g^{ij}_{(0)}\frac{\partial g^{(0)}_{ij}}{\partial T}\,\delta T_{[1]} + g^{ij}_{(0)}\frac{\partial g^{(0)}_{ij}}{\partial \bar{\mu}}\,\delta\bar{\mu}_{[1]}\right) + \frac{i\omega_{[2]}}{2} g^{ij}_{(0)}\delta\hat{g}^{(0)}_{[1]ij} - ik_i v^i_{[2]}\,,
\end{aligned}$$

$$\begin{aligned}
&\nabla_j\left(Z^{(0)}\left(i\,k^j\delta\bar{\mu}_{[2]} + \nabla^j w_{[3]} + v^j_{[3]}a_t^{(0)}\right)\right) = \\
&-\nabla_j\left(Z^{(0)}\left(i\,k^j\hat{w}_{[2]} + i\omega_{[2]}g^{jk}_{(0)}\delta\hat{a}^{(0)}_{[1]k}\right)\right) \\
&-ik_j\left(Z^{(0)}\left(i\,k^j(\delta\bar{\mu}_{[1]} + \hat{w}_{[1]}) + \nabla^j\hat{w}_{[2]} + v^j_{[2]}a_t^{(0)}\right)\right) \\
&+i\omega_{[3]}\left(\frac{1}{2}Z^{(0)}a_t^{(0)}g^{ij}_{(0)}\frac{\partial g^{(0)}_{ij}}{\partial T} + \partial_\phi Z^{(0)}a_t^{(0)}\frac{\partial \phi^{(0)}}{\partial T} + Z^{(0)}\frac{\partial a_t^{(0)}}{\partial T}\right)\delta T \\
&+i\omega_{[3]}\left(\frac{1}{2}Z^{(0)}a_t^{(0)}g^{ij}_{(0)}\frac{\partial g^{(0)}_{ij}}{\partial \bar{\mu}} + \partial_\phi Z^{(0)}a_t^{(0)}\frac{\partial \phi^{(0)}}{\partial \bar{\mu}} + Z^{(0)}\frac{\partial a_t^{(0)}}{\partial \bar{\mu}}\right)\delta\bar{\mu} \\
&+i\omega_{[2]}\left(\frac{1}{2}Z^{(0)}a_t^{(0)}g^{ij}_{(0)}\frac{\partial g^{(0)}_{ij}}{\partial T} + \partial_\phi Z^{(0)}a_t^{(0)}\frac{\partial \phi^{(0)}}{\partial T} + Z^{(0)}\frac{\partial a_t^{(0)}}{\partial T}\right)\delta T_{[1]} \\
&+i\omega_{[2]}\left(\frac{1}{2}Z^{(0)}a_t^{(0)}g^{ij}_{(0)}\frac{\partial g^{(0)}_{ij}}{\partial \bar{\mu}} + \partial_\phi Z^{(0)}a_t^{(0)}\frac{\partial \phi^{(0)}}{\partial \bar{\mu}} + Z^{(0)}\frac{\partial a_t^{(0)}}{\partial \bar{\mu}}\right)\delta\bar{\mu}_{[1]}
\end{aligned}$$

$$+ i\omega_{[2]}\, Z^{(0)} \frac{1}{2} a_t^{(0)} \left(\delta\hat{g}^{(0)}_{[1]tt} - \delta\hat{g}^{(0)}_{[1]rr} + g^{ij}_{(0)} \delta\hat{g}^{(0)}_{[1]ij} \right)$$
$$+ i\omega_{[2]}\, Z^{(0)} \left(\delta\hat{a}^{(1)}_{[1]t} + \frac{1}{4\pi T} v^i_{[1]} \partial_i a_t^{(0)} \right) + i\omega_{[2]}\, \partial_\phi Z^{(0)} a_t^{(0)}\, \delta\hat{\phi}^{(0)}_{[1]}\,, \tag{3.E.2}$$

and we recall $\omega_{[1]} = 0$. We will not explicitly write out the vector constraint equation (3.3.8) at this order, which is rather long, and in fact won't play a role in the following discussion. We now want to examine the global constraints implied by the requirement that the periodic functions $v^i_{[3]}$, $\hat{w}_{[3]}$ and $\hat{p}_{[3]}$ exist. After integrating the two equations (3.E.2) over space we obtain an inhomogeneous system of algebraic equations involving the constants $\delta T_{[1]}$, $\delta\hat{\mu}_{[1]}$, which are, so far, undetermined, as well as $\omega_{[3]}$. As we will now discuss, the parts of this system that are not homogeneous in these variables will involve integrals of functions which are fixed at first order in perturbation theory, as well as $\omega_{[2]}$ which was already fixed in the main text, following (3.4.14). As we will see, it is important to identify the implicit dependence of these two equations on $\delta T_{[1]}$ and $\delta\bar{\mu}_{[1]}$ as well as the manifest explicit dependence.

After solving the leading order constraint equations as well as the radial equations, we know that $\hat{w}_{[1]}$, $\hat{p}_{[1]}$,$v^i_{[1]}$ and indeed all the first order functions $\hat{\mathbf{\Phi}}_{[1]}$ are proportional to the constants δT and $\delta\mu$. Furthermore, the constants δT and $\delta\mu$ are not independent of each other: from (3.4.14)-(3.4.16), the existence of the perturbation at second order imposes that for each of the two diffusive modes we must have

$$\begin{pmatrix} \delta T_\pm \\ \delta\bar{\mu}_\pm \end{pmatrix} = \delta h \mathbb{V}_\pm\,, \tag{3.E.3}$$

where δh is a constant, the vector $\mathbb{V}_\pm$ belongs to the kernel of the matrix

$$\mathbb{M}_\pm = \begin{pmatrix} T^{-1}\left(i\,\omega^\pm_{[2]}\, c_\mu - \bar{\kappa}(k) \right) & i\,\omega^\pm_{[2]}\, \xi - \alpha(k) \\ i\,\omega^\pm_{[2]}\, \xi - \alpha(k) & i\,\omega^\pm_{[2]}\, \chi - \sigma(k) \end{pmatrix}\,, \tag{3.E.4}$$

and $\omega^\pm_{[2]}$ is given in (3.4.16). It is also convenient to introduce the vector $\mathbb{V}^\perp_\pm$ which is orthogonal to $\mathbb{V}_\pm$. Then there is some constants $\delta h^{||}_{[1]}$ and $\delta h^{\perp}_{[1]}$ such that we can write

$$\begin{pmatrix} \delta T_{[1]} \\ \delta\bar{\mu}_{[1]} \end{pmatrix} = \delta h^{||}_{[1]}\, \mathbb{V}_\pm + \delta h^{\perp}_{[1]}\, \mathbb{V}^\perp_\pm\,. \tag{3.E.5}$$

We stress here that the component $\delta h^{||}_{[1]}$ is redundant and we do not expect it to be fixed by the equations of motion. This follows from the fact we are examining a linearised perturbation and we should be able to freely choose $\delta h^{||}_{[1]}$ by scaling the whole solution by a function of ε. Multiplying the whole perturbation by e.g. $1+\alpha\,\varepsilon$ would effectively shift $\delta h^{||}_{[1]} \to \delta h^{||}_{[1]} + \alpha\,\delta h$. As we will see, the constant $\delta h^{||}_{[1]}$ does indeed drop out from the two algebraic equations that we obtain by integrating over (3.E.2).

Using this notation, for the first order functions $\hat{\mathbf{\Phi}}_{[1]}$, which just depend on δT,$\delta\mu$ we can write

$$\hat{\mathbf{\Phi}}_{[1]} = \hat{\mathbf{\Phi}}_{[1][0]}\,\delta h\,. \tag{3.E.6}$$

For the second order functions $\hat{\mathbf{\Phi}}_{[2]}$ the situation is more involved and we have

$$\begin{aligned}\hat{\mathbf{\Phi}}_{[2]} &= \hat{\mathbf{\Phi}}_{[2][0]}\,\delta h + \hat{\mathbf{\Phi}}_{[2]T}\,\delta T_{[1]} + \hat{\mathbf{\Phi}}_{[2]\mu}\,\delta\bar{\mu}_{[1]}\,,\\ &= \hat{\mathbf{\Phi}}_{[2][0]}\,\delta h + \hat{\mathbf{\Phi}}_{[2][1]}\,\delta h^{||}_{[1]} + \hat{\mathbf{\Phi}}^{\perp}_{[2][1]}\,\delta h^{\perp}_{[1]}\,.\end{aligned} \tag{3.E.7}$$

In particular

$$\begin{aligned}\hat{w}_{[2]} &= \hat{w}_{[2][0]}\delta h + \hat{w}_{[2]T}\,\delta T_{[1]} + \hat{w}_{[2]\mu}\,\delta\bar{\mu}_{[1]}\,,\\ \hat{p}_{[2]} &= \hat{p}_{[2][0]}\delta h + \hat{p}_{[2]T}\,\delta T_{[1]} + \hat{p}_{[2]\mu}\,\delta\bar{\mu}_{[1]}\,,\\ v^i_{[2]} &= v^i_{[2][0]}\delta h + v^i_{[2]T}\,\delta T_{[1]} + v^i_{[2]\mu}\,\delta\bar{\mu}_{[1]}\,.\end{aligned} \tag{3.E.8}$$

The parts of the solutions of these horizon quantities that are proportional to δh can be found from the constraints (3.4.13) after setting $\delta T_{[1]}$ and $\delta\bar{\mu}_{[1]}$ equal to zero. The key observation, now, is that in the constraint equations at second order (i.e. (3.4.13) as well as the vector constraint equation), the pieces in (3.E.8) proportional to $\delta T_{[1]}$ and $\delta\bar{\mu}_{[1]}$ are precisely the same equations that we have in the DC calculation outlined in Appendix 3.C with $E_i = -i\,k_i\,\delta\bar{\mu}_{[1]}$ and $\zeta_i = -i\,k_i\,\delta T_{[1]}/T$. We can therefore write

$$\begin{aligned}&4\pi T i\oint\sqrt{g_{(0)}}v^i_{[2]T} = \bar{\kappa}^{ij}_H k_j\,, \qquad 4\pi T i\oint\sqrt{g_{(0)}}v^i_{[2]\mu} = T\bar{\alpha}^{ij}_H k_j\,,\\ &i\oint\sqrt{g_{(0)}}Z^{(0)}\left(-i\,k^i + \nabla^i\hat{w}_{[2]\mu} + v^i_{[2]\mu}a^{(0)}_t\right) = \sigma^{ij}_H k_j\,,\\ &i\oint\sqrt{g_{(0)}}Z^{(0)}\left(\nabla^i\hat{w}_{[2]T} + v^i_{[2]T}a^{(0)}_t\right) = \alpha^{ij}_H k_j\,.\end{aligned} \tag{3.E.9}$$

With the ingredients assembled above, we now integrate equations (3.E.2) and find that we can write them in the form

$$\begin{aligned}&i\,\omega_{[3]}\,\mathbb{S}\mathbb{V}_{\pm}\delta h + \mathbb{M}_{\pm}\left(\mathbb{V}_{\pm}\,\delta h^{||}_{[1]} + \mathbb{V}^{\perp}_{\pm}\,\delta h^{\perp}_{[1]}\right) + \mathbb{W}\,\delta h = 0\,,\\ &\qquad\Rightarrow i\,\omega_{[3]}\,\mathbb{S}\mathbb{V}_{\pm}\delta h + \mathbb{M}_{\pm}\,\mathbb{V}^{\perp}_{\pm}\,\delta h^{\perp}_{[1]} + \mathbb{W}\,\delta h = 0\,,\end{aligned} \tag{3.E.10}$$

where we have defined the matrix of susceptibilities

$$\mathbb{S} = \begin{pmatrix} T^{-1}c_\mu & \xi \\ \xi & \chi \end{pmatrix}\,. \tag{3.E.11}$$

The vector $\mathbb{W}\delta h$ is defined through the integrals of the *functions* that appear in (3.E.2) with index [1] and also through the $v^i_{[2][0]}$ part of the horizon fluid velocity, both of which are proportional to δh. Equation (3.E.10) now fixes both $\omega_{[3]}$ as a function of k^i and $\delta h^{\perp}_{[1]}$ as a function of δh. In particular, this shows how the zero modes of $\delta T_{[1]}$, $\delta\bar{\mu}_{[1]}$ are fixed at this order of perturbation theory.

It is also clear from the above analysis, that a similar structure will repeat itself at higher orders in the perturbative expansion, fixing the zero modes of $\delta T_{[i]}$, $\delta\bar{\mu}_{[i]}$ for $i > 1$. In particular, in the expression (3.E.1) we will have $\hat{\mathbf{\Phi}}_{[i]}$ depending on $\delta T_{[i-1]}$ and $\delta\bar{\mu}_{[i-1]}$ in an analogous way.

Chapter 4

Incoherent transport for phases that spontaneously break translations

This chapter is a reproduction of [3], written in collaboration with Aristomenis Donos, Jerome Gauntlett and Tom Griffin.

In this paper, we generalise some comments made in subsection 1.2.4 to phases of matter at finite charge density which spontaneously break spatial translations. More precisely, in section 4.2, by performing a finite frequency boost we are able to show that there is only one independent element in the conductivity matrix (1.2.1). Indeed, $\sigma, \alpha, \bar{\alpha}$ and $\bar{\kappa}$ are all related, as in translationally invariant systems. We then derive expressions for the small frequency behaviour of the thermoelectric conductivities generalising those that have been derived in a translationally invariant context. We also identify a boost invariant incoherent current operator (without taking a hydrodynamic limit), which decouples from the momentum and whose conductivity is a specific finite combination of the thermoelectric conductivities.

In section 4.3 we show that, within holographic constructions, the DC conductivity for the incoherent current can be obtained from a solution to a Stokes flow for an auxiliary fluid on the black hole horizon combined with specific thermodynamic quantities associated with the equilibrium black hole solutions. Heuristically, this comes from observing that the prescription of subsection 1.2.4 fails in spontaneously modulated backgrounds due to the non-uniqueness of the bulk solution, which leads to non-uniqueness of the boundary currents and thus to infinite DC conductivities. In contrast, the incoherent current is boost invariant, and this leads to a finite, unique incoherent DC conductivity in terms of horizon data.

4.1 Introduction

Studying the thermoelectric transport properties of quantum critical states of matter at finite charge density is a topic of great theoretical and practical importance. For 'clean systems', i.e. systems that are translationally invariant and hence without a mechanism for momentum to dissipate, it is well known that the DC conductivities are infinite. More precisely, the translation invariance implies that momentum is conserved and this leads to the appearance of a delta function in the thermoelectric AC conductivities at zero frequency.

For translationally invariant systems the notion of an 'incoherent current' was introduced in [134], building on [286]. This was defined to be a linear combination of the electric and heat currents that has zero overlap with the momentum operator. Using the hydrodynamic results of [132], which implicitly assumed that the system was not in a superfluid state, it is then easy to see that the incoherent current should have finite DC conductivity, $[\sigma_{inc}]_{DC}$. In first order relativistic hydrodynamics, there is only one independent transport coefficient for both neutral systems as well as systems at finite chemical potential. In this context, the retarded two point function of the incoherent current operator provides a generalisation of the Kubo formula for that transport coefficient which is appropriate for systems at finite chemical potential. It was shown in [134] that expressions for the low frequency behaviour of the thermoelectric conductivities can be expressed in terms of $[\sigma_{inc}]_{DC}$ and certain thermodynamics quantities. Furthermore, it was also shown how $[\sigma_{inc}]_{DC}$ can be calculated within a specific class of holographic models from data at the black hole horizon.

The goal of this short paper is to generalise some of these results to phases of relativistic systems, held at finite chemical potential with respect to an abelian symmetry, that break translations spontaneously. Our general arguments, which are rather simple, will not assume any hydrodynamic limit of the system. That is, for a given temperature we will allow for phases with arbitrary spatial modulation. We will identify a universal boost-invariant incoherent current and argue that when there is no superfluid the low frequency behaviour of the thermoelectric conductivities can still be expressed in terms of certain thermodynamics quantities as well as the finite incoherent DC conductivity, $[\sigma_{inc}]_{DC}$. Within a holographic context, describing a strongly coupled system, we also explain how $[\sigma_{inc}]_{DC}$ can be calculated in terms of a Stokes flow on the spatially modulated black hole horizon, supplemented with some thermodynamic quantities of the background. This extends the results of [290] that obtained the DC conductivities for holographic systems for which the translations are explicitly broken.

Naturally, we will focus on the properties of spatially modulated phases that are thermodynamically preferred. Such phases, which may have anisotropic spatial modulation, necessarily satisfy the condition $\langle \bar{T}^{ij} \rangle = p\delta^{ij}$, where $\bar{T}^{ij}$ is the constant zero mode part of the spatial components of the stress tensor [343, 344]. However, since spatially modulated phases in which this condition is not satisfied have been analysed in a holographic context in [345] we briefly comment on some of the modified formula in appendix 4.A. In particular our general results on how to derive the $[\sigma_{inc}]_{DC}$ within holography immediately lead to the result presented in [345] for the specific holographic model studied there.

More generally, charge and spin density waves and their impact on the phenomenology of condensed matter systems have been of central interest for a long time e.g. [212]. Some more recent work on thermoelectric transport for phases that spontaneously break translations has appeared in [302], which included the effects of disorder and pinning in a hydrodynamic description, as well as in a number of holographic studies, including [340, 346, 347] and brane probe models [276, 342]. An interesting open topic, which is left for the future, would be to derive the effective hydrodynamic description of the specific examples of spontaneously formed density wave states which have already been studied within holography, along the lines of [234].

4.2 Boost invariant incoherent current

Consider a relativistic quantum field theory at finite temperature defined on flat spacetime. We will consider the system to be held at constant chemical potential, μ, with respect to an abelian global symmetry. We will also allow for the possibility for additional deformations of the Hamiltonian by a scalar operator $\mathcal{O}_\phi$ that is parametrised by the constant source ϕ_s. If $\mathcal{O}_\phi$ is odd under time reversal invariance then a non-zero ϕ_s will explicitly break time reversal invariance.

We are particularly interested in phases in which spatial translations are broken spontaneously, but our analysis will also cover translationally invariant phases. We will assume that the system reaches local thermodynamic equilibrium satisfying periodic boundary conditions generated by a set of lattice vectors $\{\mathbf{L}_i\}$. Thus, the expectation values of the stress tensor density, $\langle T^{\mu\nu} \rangle$, the conserved abelian current density, $\langle J^\mu \rangle$, as well $\langle \mathcal{O}_\phi \rangle$ are all functions of the spatial coordinates, $\mathbf{x}$, which are taken to be cartesian coordinates, that are invariant under shifts by any of the lattice vectors. For any such function, $A(\mathbf{x})$, the zero mode is denoted by $\bar{A}$, with $\bar{A} = \oint A \equiv \frac{1}{vol} \int_{\{0\}}^{\{\mathbf{L_i}\}} d\mathbf{x} A(\mathbf{x})$, where the volume of a unit cell of the lattice is $vol \equiv \int_{\{0\}}^{\{\mathbf{L_i}\}} d\mathbf{x}$.

It is important to recall that the thermodynamically preferred configurations will necessarily satisfy certain constraints on the zero modes of these expectation values [343, 344]. In particular, by ensuring that the free energy is minimised over the moduli space of spontaneously generated lattices, we must have $\langle \bar{T}^{ij}\rangle \equiv t^{ij} = p\delta^{ij}$, where p is the spatially averaged constant pressure density and is related to the free energy density, w, via $w = -p$. Defining the total charge density $\rho \equiv \langle \bar{J}^t\rangle$, the total energy density $\varepsilon \equiv -\langle \bar{T}^t{}_t\rangle$, and the total entropy density s, we also have the fundamental thermodynamic relation $Ts + \rho\mu = \varepsilon + p$. It was also shown in [343] that the zero mode of the heat current must vanish, $\langle \bar{Q}^i\rangle = 0$, where we recall that $Q^i \equiv -T^i{}_t - \mu J^i$. If the global $U(1)$ symmetry is not spontaneously broken, which will be the principle focus of this paper, by extending the arguments of [343], we can invoke invariance under large gauge transformations with gauge parameter $\Lambda = x^i\, q_i$ to argue that $\langle \bar{J}^i\rangle = 0$ as well. On the other hand for a superfluid one can have $\langle \bar{Q}^i\rangle = 0$ with $\langle \bar{J}^i\rangle \neq 0$ since a non-trivial external gauge field of the form $A_i = q_i$ cannot be gauged away, being associated with a supercurrent. However, for the thermodynamically preferred phase obtained by minimising the free energy with respect to q_i, we have once again that $\langle \bar{J}^i\rangle = 0$. We also note here that $P_{(i)} \equiv \bar{T}^t{}_i$ is the time independent charge associated with the total momentum density operator in the ith direction.

We now deduce some simple facts about the two-point functions for the current-current retarded Green's functions. These can be obtained from Ward identities, generalising [124], but we find it illuminating to obtain them by generating a time-dependent perturbation via the coordinate transformation

$$x^i \to x^i + \lambda e^{-i\omega t}\, \xi^i\,, \tag{4.2.1}$$

where λ is a small parameter and ξ^i is a constant vector. Notice that for small ω this is a translation combined with a Galilean boost. By taking the Lie derivative with respect to the vector $k^\mu = (0, \lambda e^{-i\omega t}\, \xi^i)$, we easily determine how various quantities transform. The transformed metric is $ds^2 = -dt^2 + \delta_{ij}dx^i dx^j - 2i\omega\, \lambda e^{-i\omega t}\, \xi_i dx^i dt$ and the perturbation $\delta g_{ti} = -i\omega\, \lambda e^{-i\omega t}\, \xi_i$ parametrises a source for the operator T^{ti} in the action. Equivalently, it generates[1] a spatially independent source in the Hamiltonian associated with the operator $-T^i{}_t = Q^i + \mu J^i$ and with parameter $+i\omega\lambda e^{-i\omega t}\xi_i$.

The coordinate transformation also modifies the stress tensor and current densities and we find

$$\delta T^t{}_t = \lambda e^{-i\omega t}\left(\xi^k\partial_k T^t{}_t + i\omega\xi_i T^i{}_t\right)\,, \qquad \delta T^t{}_i = \lambda e^{-i\omega t}\xi^k\partial_k T^t{}_i\,,$$

[1] It also can be viewed as generating a source in the Hamiltonian for the operator $T^t{}_i$ with parameter $+i\omega\lambda e^{-i\omega t}\xi^i$.

$$
\begin{aligned}
\delta T^i{}_t =&\lambda e^{-i\omega t}\left(\xi^k\partial_k T^i{}_t + i\omega\left[\xi^i\, T^t{}_t - \xi^j T^i{}_j\right]\right)\,,\\
\delta T^i{}_j =&\lambda e^{-i\omega t}\left(\xi^k\partial_k T^i{}_j + i\omega\xi^i T^t{}_j\right)\,, \qquad \delta\mathcal{O}_\phi = \lambda e^{-i\omega t}\,\xi^k\partial_k\mathcal{O}_\phi\,,\\
\delta J^t =&\lambda e^{-i\omega t}\,\xi^k\partial_k J^t\,, \qquad \delta J^i = \lambda e^{-i\omega t}\left(\xi^k\partial_k J^i + i\omega\xi^i J^t\right)\,.
\end{aligned}
\tag{4.2.2}
$$

Focussing now on the zero modes we have

$$
\begin{aligned}
\delta\bar{T}^t{}_t =&\lambda e^{-i\omega t} i\omega\xi_i\bar{T}^i{}_t\,, \qquad \delta\bar{T}^t{}_i = 0\,,\\
\delta\bar{T}^i{}_t =&\lambda e^{-i\omega t} i\omega\left[\xi^i\,\bar{T}^t{}_t - \xi^j\bar{T}^i{}_j\right]\,, \qquad \delta\bar{T}^i{}_j = \lambda e^{-i\omega t} i\omega\xi^i\bar{T}^t{}_j\,,\\
\delta\bar{J}^t =&0\,, \qquad \delta\bar{J}^i = \lambda e^{-i\omega t} i\omega\bar{J}^t\,\xi^i\,, \qquad \delta\bar{\mathcal{O}}_\phi = 0\,.
\end{aligned}
\tag{4.2.3}
$$

In particular, we notice from the first line that while the Hamiltonian has changed, the total momentum density operator is unchanged $\delta P_{(i)} \equiv \delta\bar{T}^t{}_i = 0$ (and it is worth highlighting that $\delta\bar{T}^i{}_t \neq 0$).

From these expressions we can immediately read off the one-point function responses of the system to the source for the operator $Q^i + \mu J^i$, with parameter $+i\omega\lambda e^{-i\omega t}\xi_i$. For example, we have

$$
\delta\langle\bar{T}^i{}_t\rangle = -\,\lambda e^{-i\omega t} i\omega\xi^i(\varepsilon + p)\,, \qquad \delta\langle\bar{J}^i\rangle = \lambda e^{-i\omega t} i\omega\rho\,\xi^i\,.
\tag{4.2.4}
$$

Hence, we can immediately deduce, in particular, that

$$
G_{J^i(Q^j+\mu J^j)}(\omega,\mathbf{0}) = \rho\,\delta^{ij}\,, \qquad G_{(Q^i+\mu J^i)(Q^j+\mu J^j)}(\omega,\mathbf{0}) = (\varepsilon+p)\delta^{ij}\,.
\tag{4.2.5}
$$

Here we are using the notation for the retarded Green's functions discussed in [1], with $G_{AB}(\omega,\mathbf{k})$ determining the zero mode linear response of an operator A to the application of a source for the operator B parametrised by a single Fourier mode labelled by $(\omega,\mathbf{k})$.

We can obtain further information using Onsager's relations, which relate Green's functions in a given background to those in a background with time-reversed sources. In the set-ups we are considering the only possible source that breaks time reversal invariance is the scalar source ϕ_s in the particular case when the operator $\mathcal{O}$ is odd under time-reversal. Thus, for example, we have in general $G_{J^iQ^j}(\omega,\mathbf{0}) = G'_{Q^jJ^i}(\omega,\mathbf{0})$, where the prime denotes the background with the opposite sign for ϕ_s. Now, suppose that we consider the time reversed background and then carry out exactly the same transformations as above. We then deduce the results (4.2.5) for the primed Green's functions, i.e. in the time reversed background, with exactly the same right hand sides (since they are inert under time-reversal): $G'_{J^i(Q^j+\mu J^j)}(\omega,\mathbf{0}) = \rho\,\delta^{ij}$ and $G'_{(Q^i+\mu J^i)(Q^j+\mu J^j)}(\omega,\mathbf{0}) = (\varepsilon+p)\delta^{ij}$. Using Onsager's relations on these expressions, and that they are explicitly symmetric in i and j, we can deduce that for the Green's functions in the original background, in addition to (4.2.5) we also have

$$G_{(Q^i+\mu J^i)J^j}(\omega,\mathbf{0}) = \rho\,\delta^{ij}\,. \tag{4.2.6}$$

As a corollary, we also have, in the original background, $G_{J^iQ^j}(\omega,\mathbf{0}) = G_{Q^iJ^j}(\omega,\mathbf{0})$.

After multiplying (4.2.5), (4.2.6) by i/ω we then deduce the following relations for the thermoelectric AC conductivity

$$\begin{aligned}\mu\,\sigma^{ij}(\omega) + T\alpha^{ij}(\omega) =& \frac{i\rho}{\omega}\,\delta^{ij}\,,\\ \mu T\bar{\alpha}^{ij}(\omega) + T\bar{\kappa}^{ij}(\omega) =& \frac{iTs}{\omega}\,\delta^{ij}\,,\end{aligned} \tag{4.2.7}$$

and we also have $\alpha^{ij}(\omega) = \bar{\alpha}^{ij}(\omega)$. The pole at $\omega = 0$ is associated with a delta function via the Kramers-Krönig relations, and is due to conservation of momentum (since any breaking of translations is assumed to be spontaneous). In the case that there is no scalar source associated with breaking of time reversal invariance, then we also have that $\sigma^{ij}(\omega)$, $\alpha^{ij}(\omega) = \bar{\alpha}^{ij}(\omega)$ and $\bar{\kappa}^{ij}(\omega)$ are all symmetric matrices.

We now define the incoherent current operator

$$J^i_{inc} \equiv (\varepsilon + p)J^i + \rho T^i{}_t = TsJ^i - \rho Q^i\,. \tag{4.2.8}$$

For the backgrounds we are considering we have $\bar{J}^i_{inc} = Ts\bar{J}^i$, which is zero both when the $U(1)$ symmetry is not spontaneously broken and also for superfluids in the thermodynamically preferred phase. We also notice that $\delta\bar{J}^i_{inc} = 0$, showing that $\bar{J}^i_{inc}$ is an invariant quantity under the finite frequency boosts (4.2.1). From (4.2.5) and (4.2.6) we have

$$G_{J^i_{inc}(Q^j+\mu J^j)}(\omega,\mathbf{0}) = G_{(Q^i+\mu J^i)J^j_{inc}}(\omega,\mathbf{0}) = 0\,. \tag{4.2.9}$$

Furthermore, defining the incoherent conductivity via $\sigma^{ij}_{inc}(\omega) \equiv \frac{i}{\omega}G_{J^i_{inc}J^j_{inc}}(\omega)$ we have

$$\sigma^{ij}_{inc}(\omega) = (Ts)^2\sigma^{ij}(\omega) - 2(Ts)\rho T\alpha^{ij}(\omega) + \rho^2 T\bar{\kappa}^{ij}(\omega)\,. \tag{4.2.10}$$

At this juncture we now assume that the $U(1)$ is unbroken (i.e. no superfluid). In this case since $\sigma^{ij}_{inc}(\omega)$ is a boost invariant quantity then we expect it to be a finite quantity at $\omega = 0$. Continuing now with this assumption it is convenient to define $[\sigma^{ij}_{inc}]_{DC} \equiv \sigma^{ij}_{inc}(\omega = 0)$ and also

$$\sigma^{ij}_0 \equiv \frac{1}{(\varepsilon + p)^2}[\sigma^{ij}_{inc}]_{DC}\,. \tag{4.2.11}$$

As we have already seen there are poles in the thermoelectric conductivity matrices, and hence associated delta functions which we suppress for the moment.

If we now assume that the analytic structure of the Green's functions is such that we can write $\sigma(\omega) \to \frac{i}{\omega}x + y$, as $\omega \to 0$, where x, y are constant matrices, then using (4.2.7) to get expressions for α and $\bar{\kappa}$ as $\omega \to 0$, as well as demanding that the pole is absent in $\sigma_{inc}(\omega)$ we immediately deduce that we can write, as $\omega \to 0$,

$$\begin{aligned}
\sigma^{ij}(\omega) &\to \left(\pi\delta(\omega) + \frac{i}{\omega}\right)\frac{\rho^2}{\varepsilon + p}\delta^{ij} + \sigma_0^{ij}\,, \\
T\bar{\alpha}^{ij} = T\alpha^{ij}(\omega) &\to \left(\pi\delta(\omega) + \frac{i}{\omega}\right)\frac{\rho T s}{\varepsilon + p}\delta^{ij} - \mu\sigma_0^{ij}\,, \\
T\bar{\kappa}^{ij}(\omega) &\to \left(\pi\delta(\omega) + \frac{i}{\omega}\right)\frac{(Ts)^2}{\varepsilon + p}\delta^{ij} + \mu^2\sigma_0^{ij}\,,
\end{aligned} \tag{4.2.12}$$

and here we have included the delta functions. This is our first main result.

Some simple corollaries now follow. We first recall that the electrical conductivity at zero total heat current can be expressed as $\sigma_{\bar{Q}=0}(\omega) \equiv \sigma(\omega) - T\alpha(\omega)\bar{\kappa}(\omega)^{-1}\bar{\alpha}(\omega)$, while the thermal conductivity at zero total electric current is given by $\kappa(\omega) \equiv \bar{\kappa}(\omega) - T\bar{\alpha}(\omega)\sigma(\omega)^{-1}\alpha(\omega)$. From (4.2.12) we deduce that $\sigma_{\bar{Q}=0}(\omega)$ and also $\kappa(\omega)$, if $\rho \neq 0$, are both finite as $\omega \to 0$ with

$$\sigma^{ij}_{\bar{Q}=0}(\omega) \to \frac{(\varepsilon + p)^2}{(Ts)^2}\sigma_0^{ij}\,, \qquad \kappa^{ij}(\omega) \to \frac{(\varepsilon + p)^2}{\rho^2}\sigma_0^{ij}\,. \tag{4.2.13}$$

Furthermore, since $\alpha^{ij}(\omega) = \bar{\alpha}^{ij}(\omega)$, we can write

$$\sigma^{ij}_{inc}(\omega) = (Ts)^2\sigma^{ij}_{\bar{Q}=0}(\omega) + [T\alpha\bar{\kappa}^{-1}\alpha(Ts - \rho\alpha^{-1}\bar{\kappa})^2]^{ij}(\omega)\,, \tag{4.2.14}$$

and we note that the second term vanishes as $\omega \to 0$.

4.3 Holography

Within holography, phases with spontaneously broken translations are described by black holes with planar horizons with a metric, gauge field and scalar which, generically, all depend periodically on all of the spatial directions. Such horizons also arise for "holographic lattices", i.e. black hole solutions which are dual to field theories that have been deformed by operators which explicitly break spatial translations.

In both cases, following [290], we briefly summarise how one can obtain the thermoelectric conductivity of the black hole horizon. To simplify the discussion we only consider background configurations that have vanishing magnetisation currents and moreover time-reversal invariance is not broken, either explicitly or spontaneously[2].

[2] More general discussions, including a careful treatment of transport currents, can be found in [293, 320].

We will also assume we are not in a superfluid phase.

One first applies a suitable DC perturbation to the full black hole solution that is linear in the time coordinate and parametrised by DC sources E_i and ζ_i, which are taken to be constant throughout the bulk spacetime. It can then be shown that on the black hole horizon a subset of the perturbation must satisfy a Stokes flow for an auxiliary fluid, with sources E_i and ζ_i. Solving these Stokes equations gives local currents on the horizon, Q^i_H and J^i_H, which depend periodically on the spatial coordinates. Determining the zero modes of these currents, denoted by $\bar{Q}^i_H$ and $\bar{J}^i_H$, and relating them to E_i and ζ_i we then obtain, by definition, the horizon DC conductivities σ^{ij}_H, α^{ij}_H, $\bar{\alpha}^{ij}_H$ and $\bar{\kappa}^{ij}_H$. In the absence of Killing vectors on the black hole horizon geometry, these will be uniquely defined and finite quantities. Since we are assuming that the background is time-reversal invariant, σ^{ij}_H and $\bar{\kappa}^{ij}_H$ are symmetric matrices and also $\alpha^{ij}_H = \bar{\alpha}^{ji}_H$.

In the case of holographic lattices, i.e. when the translations have been explicitly broken, all DC conductivities of the dual field theory will be finite and σ^{ij}_H, α^{ij}_H, $\bar{\alpha}^{ij}_H$, $\bar{\kappa}^{ij}_H$ are equal to the associated DC conductivities σ^{ij}_{DC}, α^{ij}_{DC}, $\bar{\alpha}^{ij}_{DC}$, $\bar{\kappa}^{ij}_{DC}$ of the dual field theory [290]. This result follows after showing that the zero modes of the currents on the horizon, $\bar{Q}^i_H$ and $\bar{J}^i_H$, which are finite, are equal to the zero modes of the currents at the holographic boundary.

Turning to the case that translations have been broken spontaneously, the DC conductivities of the dual field theory contain infinities due to the presence of Goldstone modes. Thus, σ^{ij}_H, α^{ij}_H, $\bar{\alpha}^{ij}_H$, $\bar{\kappa}^{ij}_H$, which are finite, are certainly not equal to the σ^{ij}_{DC}, α^{ij}_{DC}, $\bar{\alpha}^{ij}_{DC}$, $\bar{\kappa}^{ij}_{DC}$, the DC conductivities of the dual field theory. However, since the zero modes of the currents on the horizon $\bar{Q}^i_H$ and $\bar{J}^i_H$, are finite and, moreover, they are still equal to the zero modes of the currents at the holographic boundary, this seems paradoxical. The simple resolution is that the full linearised perturbation about the black hole solution, with sources parametrised by E_i and ζ_i and regular at the black hole horizon, is no longer unique in the bulk spacetime. Indeed, when translations are broken spontaneously, by carrying out a coordinate transformation of the bulk solution, we can generate additional time dependent solutions that are regular at the horizon and without additional sources at the AdS boundary, as we explain in more detail in appendix 4.B.

Nevertheless, in the case that translations are broken spontaneously we know that there is a finite DC conductivity, namely $[\sigma_{inc}]^{ij}_{DC} \equiv \sigma^{ij}_{inc}(\omega \to 0)$, and this quantity can be obtained from a Stokes flow on the horizon. One applies a DC perturbation in which we source the incoherent current, J^i_{inc}, but not the current $Q^i + \mu J^i$, and this is achieved[3] by taking $\zeta_i = -\frac{\rho}{Ts} E_i$. Solving the Stokes flow on

[3]This can by seen by writing $\tilde{J}_A = M_{AB} J_B$, where $\tilde{J}_A = (J^{inc}, Q + \mu J)$, $J_A = (J^i, Q^i)$ and

the horizon with this source, one obtains a local incoherent current on the horizon, whose zero mode is also the zero mode of the incoherent current in the boundary theory, $\bar{J}_{inc}$. Since we have $\bar{J}^i_{inc} = \Big((Ts)^2\sigma^{ij}_H - Ts\rho[T\alpha^{ij}_H + T\bar{\alpha}^{ij}_H] + \rho^2 T\bar{\kappa}^{ij}_H\Big)\, E_j/(Ts)$ we deduce that when translations are broken spontaneously[4] $[\sigma_{inc}]^{ij}_{DC}$ is given by

$$[\sigma_{inc}]^{ij}_{DC} = (Ts)^2\sigma^{ij}_H - Ts\rho[T\alpha^{ij}_H + T\bar{\alpha}^{ij}_H] + \rho^2 T\bar{\kappa}^{ij}_H\,. \tag{4.3.1}$$

In particular, we deduce that the DC conductivity for the incoherent current of the field theory can be expressed in terms of the horizon DC conductivities, obtained from the solution to the Stokes flow on the horizon, combined with specific thermodynamic quantities of the equilibrium black hole solutions, which also can be obtained from the horizon. This is the second main result of this paper.

We repeat that, in general, the individual horizon conductivities on the right hand side of (4.3.1) are not the same as those of the boundary field theory. In particular, despite that fact that from (4.2.14) we have $[\sigma_{inc}]^{ij}_{DC} = (Ts)^2\sigma^{ij}_{\bar{Q}=0}(\omega\to 0)$ we do not have, in general, $\sigma^{ij}_{\bar{Q}=0}(\omega\to 0) = \sigma^{ij}_{\bar{Q}_H=0}$, where $\sigma_{\bar{Q}_H=0} \equiv \sigma_H - T\alpha_H\bar{\kappa}_H^{-1}\bar{\alpha}_H$ is the horizon DC conductivity for vanishing zero mode of the horizon heat current.

To further clarify this point, it is illuminating to now consider the black hole horizon to be a small perturbation about a flat planar space, parametrised by a small number λ. It was shown in [45, 290] that the horizon conductivities σ^{ij}_H, α^{ij}_H, $\bar{\alpha}^{ij}_H$, $\bar{\kappa}^{ij}_H$ are of order λ^{-2} but $\sigma_{\bar{Q}_H=0}$ is of order λ^0, with order λ corrections. As we explain in appendix 4.C, by extending the results of [45, 290] we can actually deduce that

$$[\sigma_{inc}]^{ij}_{DC} = (Ts)^2\sigma^{ij}_{\bar{Q}_H=0}(\lambda) + \mathcal{O}(\lambda^2)\,, \tag{4.3.2}$$

and $[\sigma_{inc}]^{ij}_{DC} \neq (Ts)^2\sigma^{ij}_{\bar{Q}_H=0}(\lambda)$, in general. If we let T_c be the temperature for the phase transition that spontaneously breaks translations, then for temperatures just below T_c the horizon will be a small deformation away from flat space, parametrised[5] by $\lambda \sim (1-T/T_c)^{1/2}$. Since, by direct calculation as in footnote 4, the value of $[\sigma_{inc}]_{DC}$ for the translation invariant background for temperatures above T_c is the

deducing that the corresponding transformed sources are $\tilde{s} = (M^T)^{-1}s$ in order that $J^T s = \tilde{J}^T\tilde{s}$. Furthermore, if we set $\zeta_i = -\frac{\rho}{Ts}E_i$ we have $\tilde{s} = (E/(Ts), 0)$.

[4]In the case of translationally invariant backgrounds, the horizon has Killing vectors and there is not a unique solution to the Stokes equations on the horizon. Specifically, we can have v^i proportional to a Killing vector on the horizon, with p, w constant, in the notation of [45, 290]. However, this ambiguity drops out of the incoherent current on the horizon, $J^i_{Hinc} \equiv (Ts)J^i_H - \rho Q^i_H$, which in this setting is constant. Furthermore, applying sources with $\zeta_i = -\frac{\rho}{Ts}E_i$ and writing $J^i_{Hinc} = [\sigma_{inc}]^{ij}_{DC}E_j/(Ts)$ we can obtain an expression for $[\sigma_{inc}]^{ij}_{DC}$. For example, for the general class of models considered in [45, 290], we get $[\sigma_{inc}]^{ij}_{DC} = (Ts)^2\sqrt{g_0}g_0^{ij}Z_0$. This gives an alternative approach to obtaining $[\sigma_{inc}]^{ij}_{DC}$ than that discussed in [134].

[5]Here we are assuming that the phase transition has mean field exponents, with the expectation value of the order parameter proportional to $(1-T/T_c)^{1/2}$. This implies that the horizon can be expanded in the same parameter about flat space.

same as $(Ts)^2\sigma_{\bar{Q}_H=0}(\lambda \to 0)$, we see that $[\sigma_{inc}]_{DC}$ is continuous as the temperature is lowered.

We can also consider λ to parametrise a small explicit breaking of translations added to a system that spontaneously breaks translations. In this case, all of the individual thermoelectric conductivity matrices of the dual field theory are finite and equal to the horizon quantities. In this case it will only be near $T = T_c$ in which the horizon is a small deformation about flat space and then one can expand in either λ or $(1 - T/T_c)^{1/2}$.

4.A Non-thermodynamically preferred phases

If we consider the system in thermal equilibrium, but do not assume that we have minimised the action with respect to the size and shape of the spontaneously formed lattice, as in [345], then the formulas in the text are modified slightly. It is helpful to introduce the symmetric matrix m^{ij} defined by

$$m^{ij} \equiv (\varepsilon - \mu\rho)\,\delta^{ij} + t^{ij}\,, \tag{4.A.1}$$

so that for the thermodynamically preferred branches we have $m^{ij} = Ts\delta^{ij}$.

Equations (4.2.5),(4.2.6) get modified to

$$\begin{aligned} G_{J^i(Q^j+\mu J^j)}(\omega,\mathbf{0}) = G_{(Q^i+\mu J^i)J^j}(\omega,\mathbf{0}) &= \rho\,\delta^{ij}\,,\\ G_{(Q^i+\mu J^i)(Q^j+\mu J^j)}(\omega,\mathbf{0}) &= [m+\mu\rho]^{ij}\,. \end{aligned} \tag{4.A.2}$$

This implies that (4.2.7) should be changed to

$$\begin{aligned} \mu\,\sigma^{ij}(\omega) + T\alpha^{ij}(\omega) &= \frac{i\rho}{\omega}\,\delta^{ij}\,,\\ \mu T\bar{\alpha}^{ij}(\omega) + T\bar{\kappa}^{ij}(\omega) &= \frac{i}{\omega} m^{ij}\,, \end{aligned} \tag{4.A.3}$$

and $\alpha^{ij}(\omega) = \bar{\alpha}^{ij}(\omega)$. The definition of the incoherent current operator is modified to

$$J^i_{inc} \equiv [mJ]^i - \rho Q^i\,. \tag{4.A.4}$$

From (4.A.2) we have

$$G_{J^i_{inc}(Q^j+\mu J^j)}(\omega,\mathbf{0}) = G_{(Q^i+\mu J^i)J^j_{inc}}(\omega,\mathbf{0}) = 0\,, \tag{4.A.5}$$

and the incoherent conductivity, $\sigma^{ij}_{inc}(\omega) \equiv \frac{i}{\omega} G_{J^i_{inc}J^j_{inc}}(\omega)$, is given by

$$\sigma^{ij}_{inc}(\omega) = [m\sigma(\omega)m]^{ij} - \rho[T\alpha(\omega)m + mT\alpha(\omega)]^{ij} + \rho^2 T\bar{\kappa}^{ij}(\omega)\,. \tag{4.A.6}$$

Writing $\sigma(\omega) \to \frac{i}{\omega}x + y$, as $\omega \to 0$, where x, y are constant matrices, as in the text, we deduce that

$$\begin{aligned}
\sigma^{ij}(\omega) &\to \left(\pi\delta(\omega) + \frac{i}{\omega}\right)\rho^2[(m+\mu\rho)^{-1}]^{ij} + \sigma_0^{ij}\,, \\
T\bar{\alpha}^{ij} = T\alpha^{ij}(\omega) &\to \left(\pi\delta(\omega) + \frac{i}{\omega}\right)\rho[m(m+\mu\rho)^{-1}]^{ij} - \mu\sigma_0^{ij}\,, \\
T\bar{\kappa}^{ij}(\omega) &\to \left(\pi\delta(\omega) + \frac{i}{\omega}\right)[m^2(m+\mu\rho)^{-1}]^{ij} + \mu^2\sigma_0^{ij}\,,
\end{aligned} \tag{4.A.7}$$

where

$$\sigma_0^{ij} \equiv [(m+\mu\rho)^{-1}[\sigma_{inc}]_{DC}(m+\mu\rho)^{-1}]^{ij}\,, \tag{4.A.8}$$

and $[\sigma_{inc}]_{DC}^{ij} = \sigma_{inc}^{ij}(\omega = 0)$.

In the holographic setting, in order to get $[\sigma_{inc}]_{DC}^{ij}$ we can solve the Stokes flow on the horizon with the following constraint on the sources: $\zeta_i = -\rho(m^{-1}E)_i$. This leads to

$$[\sigma_{inc}]_{DC}^{ij} = [m\sigma_H m]^{ij} - \rho T[\bar{\alpha}_H m + m\alpha_H]^{ij} + \rho^2 T\bar{\kappa}_H^{ij}\,. \tag{4.A.9}$$

Once again we can obtain $[\sigma_{inc}]_{DC}^{ij}$ from horizon data supplemented with thermodynamic properties of the background. It is worth noting that here, in contrast to (4.3.1), not all of the thermodynamic quantities can be obtained directly from the horizon.

The above formulae simplify somewhat for the special case of spatially isotropic phases in which all of the horizon conductivities are proportional to the identity matrix and furthermore $t^{ij} = t\delta^{ij}$, so that $m^{ij} = (Ts + w + t)\delta^{ij}$. In this setting we have

$$\begin{aligned}
\sigma(\omega) &\to \left(\pi\delta(\omega) + \frac{i}{\omega}\right)\frac{\rho^2}{\varepsilon + t} + \sigma_0\,, \\
T\bar{\alpha} = T\alpha(\omega) &\to \left(\pi\delta(\omega) + \frac{i}{\omega}\right)\frac{\rho(Ts + w + t)}{\varepsilon + t} - \mu\sigma_0\,, \\
T\bar{\kappa}(\omega) &\to \left(\pi\delta(\omega) + \frac{i}{\omega}\right)\frac{(Ts + w + t)^2}{\varepsilon + t} + \mu^2\sigma_0\,,
\end{aligned} \tag{4.A.10}$$

where

$$\sigma_0 = \frac{1}{(\varepsilon + t)^2}[\sigma_{inc}]_{DC}\,. \tag{4.A.11}$$

In the holographic setting, for this special case, we can write

$$[\sigma_{inc}]_{DC} = (Ts + w + t)^2\sigma_{\bar{Q}_H=0} + T\alpha^2\bar{\kappa}^{-1}(Ts + w + t - \rho\alpha^{-1}\bar{\kappa})^2\,. \tag{4.A.12}$$

For the special case of an isotropic Q-lattice with d spatial dimensions we can be more explicit and this will allow us to recover the result of [345] for $[\sigma_{inc}]_{DC}$ who used a different approach. Using the same notation as in section 4.1 of [45] the breaking of translations is specified by a matrix $\mathcal{D}_{ij}$, which for an isotropic lattice can be written as $\mathcal{D}_{ij} \equiv \mathcal{D}\delta_{ij}$. Substituting the results of [45] into (4.A.12) then easily gives

$$[\sigma_{inc}]_{DC} = (Ts + w + t)^2 (\frac{s}{4\pi})^{(d-2)/d} Z_H + \frac{4\pi\rho^2 (w+t)^2}{s\mathcal{D}} . \tag{4.A.13}$$

Combining this with (4.A.11) and setting $d = 2$, we obtain equation (74) of [345] after identifying $w + t$ with $-2K$ in their notation.

4.B Bulk non-uniqueness

We consider a holographic theory describing a relativistic quantum field theory at finite temperature defined on flat spacetime. The system is held at constant chemical potential, μ, with respect to an abelian global symmetry and we will also allow for the possibility for additional deformations of the Hamiltonian by an uncharged scalar operator $\mathcal{O}_\phi$ that is parameterised by the constant ϕ_s.

We consider the following bulk coordinate transformations

$$x^i \to x^i - u^i(t + S(r)) , \qquad t \to t - v_i x^i , \tag{4.B.1}$$

as well as a gauge transformation with parameter $\Lambda = \mu w_i x^i$, where u^i, v_i and w_i are all constant vectors. Here $S(r)$ is a function of the holographic radial coordinate such that $S(r) = \frac{\ln r}{4\pi T} + ...$ near the horizon, located at $r \to 0$, and $S(r) \to 0$ as one approaches the AdS boundary located at $r \to \infty$. This transformation adds the following boundary sources:

$$\delta g_{ti} = v_i - \delta_{ij} u^j , \qquad \delta A_i = \mu(w_i - v_i) . \tag{4.B.2}$$

In particular, setting $v_i = \delta_{ij} u^j$ and $w_i = v_i$ gives a source free transformation that is regular at the black hole horizon. This means that, demanding a given set of sources on the AdS boundary combined with regularity at the black hole horizon, does not lead to a unique solution to the bulk equations of motion.

If we take the parameters to be infinitesimal perturbations we also deduce the following transformations on the currents in the boundary field theory:

$$\begin{aligned} \delta\langle J^i \rangle &= - t u^k \partial_k \langle J^i \rangle + u^i \langle J^t \rangle , \\ \delta\langle T^i{}_t \rangle &= - t u^k \partial_k \langle T^i{}_t \rangle + u^i \langle T^t{}_t \rangle - u^j \langle T^i{}_j \rangle . \end{aligned} \tag{4.B.3}$$

If we consider the zero modes we have

$$\begin{aligned}\delta\langle\bar{J}^i\rangle =& u^i\rho\,,\\ \delta\langle\bar{T}^i{}_t\rangle =& -u^j(\varepsilon\delta^i_j+t^{ik}\delta_{jk})\,,\end{aligned}\tag{4.B.4}$$

and also $\delta\langle\bar{Q}^i\rangle=-\delta\langle\bar{T}^i{}_t\rangle-\mu\delta\langle\bar{J}^i\rangle=u^jm^{ik}\delta_{jk}$.[6] We see that both $\langle\bar{J}^i\rangle$ and $\langle\bar{Q}^i\rangle$ are changed by this transformation (when $\rho\neq 0$). In particular, this means that the DC thermoelectric conductivity matrix is not well defined (when $\rho\neq 0$). Note, however, that $\delta\langle\bar{J}^i_{inc}\rangle=0$.

This non-uniqueness of bulk solutions (and of the currents $\langle\bar{J}^i\rangle$ and $\langle\bar{Q}^i\rangle$) means that we must be careful when calculating the DC conductivities. A solution to the perturbed equations parametrised by E_i,ζ_i may be found for which the DC current response is finite everywhere (including at the horizon and at the boundary), but when this is not the unique solution for the current, the associated DC conductivity will not be well-defined. The conclusion is that when calculating DC conductivities, it is important to first establish that the associated current response is uniquely defined.

4.C Perturbative Lattice

We follow the analysis and notation of [45, 290] which focussed on Einstein-Maxwell-dilaton theory with Lagrangian density $\mathcal{L}=R-V(\phi)-\frac{1}{4}Z(\phi)F^2-\frac{1}{2}(\partial\phi)^2$. For the black holes of interest, which preserve time reversal invariance, we assume that at the black hole horizon we can expand about a flat geometry using a perturbative parameter λ:

$$\begin{aligned}g_{(0)ij}&=g\,\delta_{ij}+\lambda\,h^{(1)}_{ij}+\cdots\,, & Z^{(0)}a^{(0)}_t&=a+\lambda\,a_{(1)}+\cdots\,,\\ \phi^{(0)}&=\psi_{(0)}+\lambda\,\psi_{(1)}+\cdots\,, & Z^{(0)}&=z_{(0)}+\lambda\,z_{(1)}+\cdots\,,\end{aligned}\tag{4.C.1}$$

with a, $z_{(0)}$, $\psi_{(0)}$ and g being constant and the sub-leading terms are functions of, generically, all of the spatial coordinates x^i and they respect the lattice symmetry. We can calculate the entropy density $s=\oint s_H$ and the charge density $\rho=\oint\rho_H$ on the horizon using

$$\begin{aligned}s_H&\equiv 4\pi\sqrt{g_{(0)}}=4\pi g^{d/2}(1+\lambda\frac{h^{(1)}}{2g}+\cdots),\\ \rho_H&\equiv\sqrt{g_{(0)}}Z^{(0)}a^{(0)}_t=ag^{d/2}(1+\lambda(\frac{h^{(1)}}{2g}+\frac{a_{(1)}}{a})+\cdots)\,,\end{aligned}\tag{4.C.2}$$

[6] Note that on the thermodynamically preferred branch we have $\delta\langle\bar{J}^i\rangle=u^i\rho$, $\delta\langle\bar{T}^i{}_t\rangle=-u^i(\varepsilon+p)$ and $\delta\bar{Q}^i=u^iTs$

where $h^{(1)} = \delta^{ij} h^{(1)}_{ij}$ and d is the number of spatial dimensions. We thus[7] have $s = 4\pi g^{d/2} + \mathcal{O}(\lambda)$ and $\rho = a g^{d/2} + \mathcal{O}(\lambda)$.

As shown in [45, 290], we can solve the horizon constraint equations perturbatively in λ using the following expansion:

$$\begin{aligned} v^i &= \frac{1}{\lambda^2}\, v^i_{(0)} + \frac{1}{\lambda}\, v^i_{(1)} + v^i_{(2)} + \cdots\,, \qquad w = \frac{1}{\lambda}\, w_{(1)} + w_{(2)} + \cdots\,, \\ p &= \frac{1}{\lambda}\, p_{(1)} + p_{(2)} + \cdots\,, \end{aligned} \tag{4.C.3}$$

where $v^i_{(0)}$ is constant. In [45, 290], this then yields a solution for the horizon DC thermoelectric conductivities σ^{ij}_H, α^{ij}_H, $\bar{\alpha}^{ij}_H$ and $\bar{\kappa}^{ij}_H$, which all have leading order behaviour of order $1/\lambda^2$.

We can now make some additional observations. We can calculate the zero modes of the electric and heat current as follows.

$$\begin{aligned} \bar{J}^i_{(0)} &\equiv \oint \sqrt{g_{(0)}} Z^{(0)} (a_t^{(0)} v^i + g^{ij}_{(0)} (\partial_j w + E_j)) = \oint \rho_H v^i + \mathcal{O}(\lambda^0)\,, \\ &= \oint \rho_H \big(\frac{1}{\lambda^2}\, v^i_{(0)} + \frac{1}{\lambda}\, v^i_{(1)}\big) + \mathcal{O}(\lambda^0) = \big(\frac{1}{\lambda^2}\, \rho v^i_{(0)} + \frac{1}{\lambda}\, \rho \bar{v}^i_{(1)}\big) + \mathcal{O}(\lambda^0)\,, \\ &= \rho \bar{v}^i + \mathcal{O}(\lambda^0)\,. \end{aligned} \tag{4.C.4}$$

Similarly,

$$\begin{aligned} \bar{Q}^i_{(0)} &\equiv 4\pi T \oint \sqrt{g_{(0)}} v^i\,, \\ &= T s \bar{v}^i + \mathcal{O}(\lambda^0)\,. \end{aligned} \tag{4.C.5}$$

We thus have $\rho \bar{Q}^i_{(0)} = sT \bar{J}^i_{(0)} + \mathcal{O}(\lambda^0)$, which means that $\rho T(\bar{\kappa}^{ij}_H \zeta_j + \bar{\alpha}^{ij}_H E_j) = sT(\sigma^{ij}_H E_j + T\alpha^{ij}_H \zeta_j) + \mathcal{O}(\lambda^0)$. Since this holds for arbitrary E_j and ζ_j, we must have:

$$\begin{aligned} \sigma^{ij}_H &= \frac{\rho}{s} \bar{\alpha}^{ij}_H + \mathcal{O}(\lambda^0)\,, \\ \alpha^{ij}_H &= \frac{\rho}{sT} \bar{\kappa}^{ij}_H + \mathcal{O}(\lambda^0)\,. \end{aligned} \tag{4.C.6}$$

The Onsager relations for this time-reversal invariant background imply that $\alpha^{ij}_H = \bar{\alpha}^{ji}_H$, $\sigma^{ij}_H = \sigma^{ji}_H$ and $\bar{\kappa}^{ij}_H = \bar{\kappa}^{ji}_H$, and so:

$$\begin{aligned} \sigma^{ij}_H &= \frac{\rho^2}{s^2 T} \bar{\kappa}^{ij}_H + \mathcal{O}(\lambda^0)\,, \\ \alpha^{ij}_H &= \bar{\alpha}^{ij}_H + \mathcal{O}(\lambda^0) = \frac{\rho}{sT} \bar{\kappa}^{ij}_H + \mathcal{O}(\lambda^0)\,. \end{aligned} \tag{4.C.7}$$

[7] Note that if preferred, one could absorb the zero modes of all the sub-leading terms in (4.C.1) into the leading terms, g, a, etc. and then s and ρ could be expressed in terms of the resummed, constant, leading terms plus corrections that would be of order $\mathcal{O}(\lambda^2)$ (since, as we see from (4.C.2), the $\mathcal{O}(\lambda)$ pieces would vanish when integrated over the spatial coordinates).

We can use this to confirm that

$$\begin{aligned}\sigma_{\bar{Q}_H=0} &\equiv \sigma_H - T\alpha_H\bar{\kappa}_H^{-1}\bar{\alpha}_H = \mathcal{O}(\lambda^0)\,,\\ \kappa_H &\equiv \bar{\kappa}_H - T\bar{\alpha}_H\sigma_H^{-1}\alpha_H = \mathcal{O}(\lambda^0)\,.\end{aligned} \tag{4.C.8}$$

Furthermore, using (4.C.7) we find

$$\rho\bar{\alpha}_H^{-1}\bar{\kappa}_H = sT + \mathcal{O}(\lambda^2)\,, \qquad \rho\bar{\kappa}_H\alpha_H^{-1} = sT + \mathcal{O}(\lambda^2)\,. \tag{4.C.9}$$

Now, from (4.3.1) we can write the DC conductivity for the incoherent current as

$$\begin{aligned}[\sigma_{inc}]_{DC} = (Ts)^2\sigma_{\bar{Q}_H=0} &+ \frac{1}{2}T\alpha_H\bar{\kappa}_H^{-1}\bar{\alpha}_H(Ts - \rho\bar{\alpha}_H^{-1}\bar{\kappa}_H)^2\\ &+ \frac{1}{2}T(Ts - \rho\bar{\kappa}_H\alpha_H^{-1})^2\alpha_H\bar{\kappa}_H^{-1}\bar{\alpha}_H\\ &+ \frac{1}{2}T\rho^2(\bar{\alpha}_H - \alpha_H)\bar{\alpha}_H^{-1}\bar{\kappa}_H - \frac{1}{2}T\rho^2\bar{\kappa}_H\alpha_H^{-1}(\bar{\alpha}_H - \alpha_H)\,.\end{aligned} \tag{4.C.10}$$

The above results then allow us to conclude that

$$[\sigma_{inc}]_{DC} = (Ts)^2\sigma_{\bar{Q}_H=0}(\lambda) + \mathcal{O}(\lambda^2)\,. \tag{4.C.11}$$

Note that we don't expect that (4.C.6) will continue to hold for higher orders in λ (it arises from the very special form of the last line in (4.C.4) and (4.C.5)). Thus, in general, $[\sigma_{inc}]_{DC} \neq (Ts)^2\sigma_{\bar{Q}_H=0}$.

Chapter 5

Outlook

The work presented in this thesis is only a small step towards a better understanding of transport at strong coupling and the relation to black hole dynamics. There are still various outstanding questions to be answered and interesting challenges to be met; here we will briefly discuss a selection of them.

The robustness and universality of the techniques of chapters 2 and 3 suggest that they can also be useful in studying systems with spontaneously broken translations or spontaneously broken global symmetries. In particular, it is possible that the formalism presented in chapter 2 can also generalise the results of [131] to the case of superfluids. Additionally, the techniques developed in chapter 3 can contribute in the clarification of the low energy excitations in holographic superfluids, which include second sound [348, 349]. In more complicated models of holographic superfluidity there are Goldstone bosons with exotic, diffusion type dispersion relations [350].

The breaking of spacetime symmetries is much less understood; for example, even the counting of the massless Goldstone bosons is not yet clear [351–353]. An intriguing prospect, currently under investigation, is the construction of Goldstone modes corresponding to spontaneously broken translational symmetry. According to the intuition gained, we expect that the near horizon region will encode the dissipative aspects of these modes, while the thermodynamic aspects will be related to the elasticity of the lattice [110, 341, 354]. Other avenues to explore would be the incorporation of magnetisation as well as higher derivative effects, which could in principle modify the generalised Einstein relations (but see [318]).

The above work can also be useful in the quest for universal bounds on diffusion [231]. In section 3.1 we discussed the growing evidence in favour of the saturation of this bound for the thermal diffusivity and the butterfly velocity in holographic theories. Specifically, in the homogeneous, momentum relaxing models considered in [106], it was crucial that both the thermal conductivity and the specific heat could be expressed in terms of the metric in the near horizon region. The calculation of the butterfly velocity also involves only the near horizon region. The explicit construction of the diffusive modes presented in chapter 3 will hopefully shed some light in these exciting connections. It could also prove to be useful in rigorous proofs of bounds on diffusion.

Apart from dissipative transport, the near horizon region can also capture significant aspects of QPTs [221]. Even more generally, it would be very interesting to explore the structure of the phase diagrams of holographic theories [355–358]. In order to make progress in this direction, it is necessary to construct various branches of black hole solutions and study their thermodynamic and dynamical properties, so that one can examine holographic phase transitions. This will lead to improved understanding of the phase diagrams of a wide range of physical systems at finite temperature or finite charge density, from strongly coupled quantum matter and high-T_c superconductors to QCD. On a related note, the construction of such backreacted black hole solutions can reveal interesting RG flows and novel ground states. There is a rich structure even in neutral, translationally invariant cases [359, 360]; the addition of spatial modulation may give rise to exotic phenomena, such as "Boomerang" flows [325, 361].

Moreover, it is worth mentioning that holography allows us to move beyond the regime of linear response and study far from equilibrium dynamical evolution of quantum systems; for instance, black hole formation as described by the Vaidya-AdS geometry is supposed to capture the essential features of thermalization after a quantum quench, see [4, 362–366] among numerous other works. Generalising to global inhomogeneous quenches [367, 368] would also be particularly exciting, in view of applications to relativistic heavy ion collisions.

Finally, recall that a characteristic of quantum matter is long-range quantum entanglement. It is intriguing that even though the calculation of entanglement entropy in many-body quantum systems and QFTs is generally hard, holography geometrises this quantity in a very elegant manner [362, 369]. This has led to a widespread use of entanglement entropy as a probe for thermalization, as well as an order parameter of quantum phase transitions [370]. There are indications that the spread of entanglement in out of equilibrium systems is related to the spread of chaos [371, 372]; it would be very interesting to investigate further such potentially profound connections.

We conclude this thesis by reiterating that even though an experimental realization of a holographic system has yet to be discovered, holography is arguably one of the most powerful tool at our disposal for gaining useful insights into strongly correlated quantum systems. The fruitful interaction between fundamental physics and condensed matter theory will undoubtedly help us to further unravel the mysteries of Nature, from quantum matter to quantum gravity.

Bibliography

[1] Aristomenis Donos, Jerome P. Gauntlett and Vaios Ziogas. 'Diffusion in inhomogeneous media'. *Phys. Rev.* D96.12 (2017), p. 125003. doi:10.1103/PhysRevD.96.125003. arXiv:1708.05412 [hep-th].

[2] Aristomenis Donos, Jerome P. Gauntlett and Vaios Ziogas. 'Diffusion for Holographic Lattices'. *JHEP* 03 (2018), p. 056. doi:10.1007/JHEP03(2018)056. arXiv:1710.04221 [hep-th].

[3] Aristomenis Donos, Jerome P. Gauntlett, Tom Griffin and Vaios Ziogas. 'Incoherent transport for phases that spontaneously break translations'. *JHEP* 04 (2018), p. 053. doi:10.1007/JHEP04(2018)053. arXiv:1801.09084 [hep-th].

[4] Vaios Ziogas. 'Holographic mutual information in global Vaidya-BTZ spacetime'. *JHEP* 09 (2015), p. 114. doi:10.1007/JHEP09(2015)114. arXiv:1507.00306 [hep-th].

[5] K. C. Cole. *A Theory of Everything.* 1987. https://www.nytimes.com/1987/10/18/magazine/a-theory-of-everything.html (visited on 01/04/2018).

[6] Elias Kiritsis. *String theory in a nutshell.* Princeton University Press, 2007.

[7] Juan Martin Maldacena. 'The Large N limit of superconformal field theories and supergravity'. *Int. J. Theor. Phys.* 38 (1999). [Adv. Theor. Math. Phys.2,231(1998)], pp. 1113–1133. doi:10.1023/A:1026654312961,10.4310/ATMP.1998.v2.n2.a1. arXiv:hep-th/9711200 [hep-th].

[8] Edward Witten. 'Anti-de Sitter space and holography'. *Adv. Theor. Math. Phys.* 2 (1998), pp. 253–291. doi:10.4310/ATMP.1998.v2.n2.a2. arXiv:hep-th/9802150 [hep-th].

[9] S. S. Gubser, Igor R. Klebanov and Alexander M. Polyakov. 'Gauge theory correlators from noncritical string theory'. *Phys. Lett.* B428 (1998), pp. 105–114. doi:10.1016/S0370-2693(98)00377-3. arXiv:hep-th/9802109 [hep-th].

[10] Gerard 't Hooft. 'Dimensional reduction in quantum gravity'. *Conf. Proc.* C930308 (1993), pp. 284–296. arXiv:gr-qc/9310026 [gr-qc].

[11] Leonard Susskind. 'The World as a hologram'. *J. Math. Phys.* 36 (1995), pp. 6377–6396. doi:10.1063/1.531249. arXiv:hep-th/9409089 [hep-th].

[12] Ofer Aharony, Steven S. Gubser, Juan Martin Maldacena, Hirosi Ooguri and Yaron Oz. 'Large N field theories, string theory and gravity'. *Phys. Rept.* 323 (2000), pp. 183–386. doi:10.1016/S0370-1573(99)00083-6. arXiv:hep-th/9905111 [hep-th].

[13] Eric D'Hoker and Daniel Z. Freedman. 'Supersymmetric gauge theories and the AdS / CFT correspondence'. *Strings, Branes and Extra Dimensions: TASI 2001: Proceedings.* 2002, pp. 3–158. arXiv:hep-th/0201253 [hep-th].

[14] Joseph Polchinski. 'Introduction to Gauge/Gravity Duality'. *Proceedings, Theoretical Advanced Study Institute in Elementary Particle Physics (TASI 2010). String Theory and Its Applications: From meV to the Planck Scale: Boulder, Colorado, USA, June 1-25, 2010.* 2010, pp. 3–46. http://inspirehep.net/record/875017/files/arXiv:1010.6134.pdfdoi:10.1142/9789814350525_0001. arXiv:1010.6134 [hep-th].

[15] Michael B. Green, J. H. Schwarz and Edward Witten. *SUPERSTRING THEORY. VOL. 1: INTRODUCTION.* Cambridge Monographs on Mathematical Physics. 1988.

[16] Michael B. Green, J. H. Schwarz and Edward Witten. *SUPERSTRING THEORY. VOL. 2: LOOP AMPLITUDES, ANOMALIES AND PHENOMENOLOGY.* 1988.

[17] J. Polchinski. *String theory. Vol. 1: An introduction to the bosonic string.* Cambridge University Press, 2007.

[18] J. Polchinski. *String theory. Vol. 2: Superstring theory and beyond.* Cambridge University Press, 2007.

[19] Gerard 't Hooft. 'A Planar Diagram Theory for Strong Interactions'. *Nucl. Phys.* B72 (1974). [,337(1973)], p. 461. doi:10.1016/0550-3213(74)90154-0.

[20] Ofer Aharony, Oren Bergman, Daniel Louis Jafferis and Juan Maldacena. 'N=6 superconformal Chern-Simons-matter theories, M2-branes and their gravity duals'. *JHEP* 10 (2008), p. 091. doi:10.1088/1126-6708/2008/10/091. arXiv:0806.1218 [hep-th].

[21] Sergio Benvenuti, Sebastian Franco, Amihay Hanany, Dario Martelli and James Sparks. 'An Infinite family of superconformal quiver gauge theories with Sasaki-Einstein duals'. *JHEP* 06 (2005), p. 064. `doi:10.1088/1126-6708/2005/06/064`. `arXiv:hep-th/0411264 [hep-th]`.

[22] Dario Martelli and James Sparks. 'Moduli spaces of Chern-Simons quiver gauge theories and AdS(4)/CFT(3)'. *Phys. Rev.* D78 (2008), p. 126005. `doi:10.1103/PhysRevD.78.126005`. `arXiv:0808.0912 [hep-th]`.

[23] Steven S. Gubser. 'Einstein manifolds and conformal field theories'. *Phys. Rev.* D59 (1999), p. 025006. `doi:10.1103/PhysRevD.59.025006`. `arXiv:hep-th/9807164 [hep-th]`.

[24] H. J. Kim, L. J. Romans and P. van Nieuwenhuizen. 'The Mass Spectrum of Chiral N=2 D=10 Supergravity on S**5'. *Phys. Rev.* D32 (1985), p. 389. `doi:10.1103/PhysRevD.32.389`.

[25] Juan Martin Maldacena and Andrew Strominger. 'AdS(3) black holes and a stringy exclusion principle'. *JHEP* 12 (1998), p. 005. `doi:10.1088/1126-6708/1998/12/005`. `arXiv:hep-th/9804085 [hep-th]`.

[26] John McGreevy, Leonard Susskind and Nicolaos Toumbas. 'Invasion of the giant gravitons from Anti-de Sitter space'. *JHEP* 06 (2000), p. 008. `doi:10.1088/1126-6708/2000/06/008`. `arXiv:hep-th/0003075 [hep-th]`.

[27] Tom Banks and Nathan Seiberg. 'Symmetries and Strings in Field Theory and Gravity'. *Phys. Rev.* D83 (2011), p. 084019. `doi:10.1103/PhysRevD.83.084019`. `arXiv:1011.5120 [hep-th]`.

[28] J. David Brown and M. Henneaux. 'Central Charges in the Canonical Realization of Asymptotic Symmetries: An Example from Three-Dimensional Gravity'. *Commun. Math. Phys.* 104 (1986), pp. 207–226. `doi:10.1007/BF01211590`.

[29] J. David Brown and James W. York Jr. 'Quasilocal energy and conserved charges derived from the gravitational action'. *Phys. Rev.* D47 (1993), pp. 1407–1419. `doi:10.1103/PhysRevD.47.1407`. `arXiv:gr-qc/9209012 [gr-qc]`.

[30] Jared Kaplan. *Lectures on AdS/CFT from the Bottom Up.* `http://sites.krieger.jhu.edu/jared-kaplan/files/2016/05/AdSCFTCourseNotesCurrentPublic.pdf` (visited on 01/04/2018).

[31] Edward Witten. 'Anti-de Sitter space, thermal phase transition, and confinement in gauge theories'. *Adv. Theor. Math. Phys.* 2 (1998). [,89(1998)], pp. 505–532. doi:10.4310/ATMP.1998.v2.n3.a3. arXiv:hep-th/9803131 [hep-th].

[32] Juan Martin Maldacena. 'Eternal black holes in anti-de Sitter'. *JHEP* 04 (2003), p. 021. doi:10.1088/1126-6708/2003/04/021. arXiv:hep-th/0106112 [hep-th].

[33] Leonard Susskind and Edward Witten. 'The Holographic bound in anti-de Sitter space' (1998). arXiv:hep-th/9805114 [hep-th].

[34] Amanda W. Peet and Joseph Polchinski. 'UV / IR relations in AdS dynamics'. *Phys. Rev.* D59 (1999), p. 065011. doi:10.1103/PhysRevD.59.065011. arXiv:hep-th/9809022 [hep-th].

[35] D. Z. Freedman, S. S. Gubser, K. Pilch and N. P. Warner. 'Renormalization group flows from holography supersymmetry and a c theorem'. *Adv. Theor. Math. Phys.* 3 (1999), pp. 363–417. doi:10.4310/ATMP.1999.v3.n2.a7. arXiv:hep-th/9904017 [hep-th].

[36] Sebastian de Haro, Sergey N. Solodukhin and Kostas Skenderis. 'Holographic reconstruction of space-time and renormalization in the AdS / CFT correspondence'. *Commun.Math.Phys.* 217 (2001), pp. 595–622. doi:10.1007/s002200100381. arXiv:hep-th/0002230 [hep-th].

[37] Vijay Balasubramanian, Per Kraus and Albion E. Lawrence. 'Bulk versus boundary dynamics in anti-de Sitter space-time'. *Phys. Rev.* D59 (1999), p. 046003. doi:10.1103/PhysRevD.59.046003. arXiv:hep-th/9805171 [hep-th].

[38] Vijay Balasubramanian and Per Kraus. 'A stress tensor for anti-de Sitter gravity'. *Commun. Math. Phys.* 208 (1999), pp. 413–428. doi:10.1007/s002200050764. arXiv:hep-th/9902121.

[39] Igor R. Klebanov and Edward Witten. 'AdS / CFT correspondence and symmetry breaking'. *Nucl. Phys.* B556 (1999), pp. 89–114. doi:10.1016/S0550-3213(99)00387-9. arXiv:hep-th/9905104 [hep-th].

[40] Ioannis Papadimitriou and Kostas Skenderis. 'Thermodynamics of asymptotically locally AdS spacetimes'. *JHEP* 0508 (2005), p. 004. doi:10.1088/1126-6708/2005/08/004. arXiv:hep-th/0505190 [hep-th].

[41] Peter Breitenlohner and Daniel Z. Freedman. 'Stability in Gauged Extended Supergravity'. *Annals Phys.* 144 (1982), p. 249. doi:10.1016/0003-4916(82)90116-6.

[42] Peter Breitenlohner and Daniel Z. Freedman. 'Positive Energy in anti-De Sitter Backgrounds and Gauged Extended Supergravity'. *Phys. Lett.* 115B (1982), pp. 197–201. `doi:10.1016/0370-2693(82)90643-8`.

[43] Ioannis Papadimitriou. 'Lectures on Holographic Renormalization'. *Springer Proc. Phys.* 176 (2016), pp. 131–181. `doi:10.1007/978-3-319-31352-8_4`.

[44] Robert M. Wald. *General Relativity*. Chicago, USA: Chicago Univ. Pr., 1984. `doi:10.7208/chicago/9780226870373.001.0001`.

[45] Elliot Banks, Aristomenis Donos and Jerome P. Gauntlett. 'Thermoelectric DC conductivities and Stokes flows on black hole horizons'. *JHEP* 10 (2015), p. 103. `doi:10.1007/JHEP10(2015)103`. `arXiv:1507.00234 [hep-th]`.

[46] Jan de Boer, Erik P. Verlinde and Herman L. Verlinde. 'On the holographic renormalization group'. *JHEP* 08 (2000), p. 003. `doi:10.1088/1126-6708/2000/08/003`. `arXiv:hep-th/9912012 [hep-th]`.

[47] Massimo Bianchi, Daniel Z. Freedman and Kostas Skenderis. 'Holographic renormalization'. *Nucl. Phys.* B631 (2002), pp. 159–194. `doi:10.1016/S0550-3213(02)00179-7`. `arXiv:hep-th/0112119 [hep-th]`.

[48] Dario Martelli and Wolfgang Mueck. 'Holographic renormalization and Ward identities with the Hamilton-Jacobi method'. *Nucl. Phys.* B654 (2003), pp. 248–276. `doi:10.1016/S0550-3213(03)00060-9`. `arXiv:hep-th/0205061 [hep-th]`.

[49] Richard L. Arnowitt, Stanley Deser and Charles W. Misner. 'Canonical variables for general relativity'. *Phys. Rev.* 117 (1960), pp. 1595–1602. `doi:10.1103/PhysRev.117.1595`.

[50] G. W. Gibbons and S. W. Hawking. 'Action Integrals and Partition Functions in Quantum Gravity'. *Phys. Rev.* D15 (1977), pp. 2752–2756. `doi:10.1103/PhysRevD.15.2752`.

[51] Steven Corley. 'A Note on holographic Ward identities'. *Phys. Lett.* B484 (2000), pp. 141–148. `doi:10.1016/S0370-2693(00)00633-X`. `arXiv:hep-th/0004030 [hep-th]`.

[52] M. Henningson and K. Skenderis. 'The Holographic Weyl anomaly'. *JHEP* 9807 (1998), p. 023. `doi:10.1088/1126-6708/1998/07/023`. `arXiv:hep-th/9806087 [hep-th]`.

[53] Ioannis Papadimitriou and Kostas Skenderis. 'AdS / CFT correspondence and geometry'. *IRMA Lect. Math. Theor. Phys.* 8 (2005), pp. 73–101. `doi:10.4171/013-1/4`. `arXiv:hep-th/0404176 [hep-th]`.

[54] Kostas Skenderis. 'Lecture notes on holographic renormalization'. *Class.Quant.Grav.* 19 (2002), pp. 5849–5876. doi:10.1088/0264-9381/19/22/306. arXiv:hep-th/0209067 [hep-th].

[55] Ioannis Papadimitriou. 'Holographic renormalization as a canonical transformation'. *JHEP* 11 (2010), p. 014. doi:10.1007/JHEP11(2010)014. arXiv:1007.4592 [hep-th].

[56] Ioannis Papadimitriou. 'Holographic Renormalization of general dilaton-axion gravity'. *JHEP* 08 (2011), p. 119. doi:10.1007/JHEP08(2011)119. arXiv:1106.4826 [hep-th].

[57] Brian P. Dolan. 'Symplectic geometry and Hamiltonian flow of the renormalization group equation'. *Int. J. Mod. Phys.* A10 (1995), pp. 2703–2732. doi:10.1142/S0217751X95001273. arXiv:hep-th/9406061 [hep-th].

[58] Thomas Faulkner, Hong Liu and Mukund Rangamani. 'Integrating out geometry: Holographic Wilsonian RG and the membrane paradigm'. *JHEP* 08 (2011), p. 051. doi:10.1007/JHEP08(2011)051. arXiv:1010.4036 [hep-th].

[59] Idse Heemskerk and Joseph Polchinski. 'Holographic and Wilsonian Renormalization Groups'. *JHEP* 06 (2011), p. 031. doi:10.1007/JHEP06(2011)031. arXiv:1010.1264 [hep-th].

[60] Andrew Chamblin, Roberto Emparan, Clifford V. Johnson and Robert C. Myers. 'Charged AdS black holes and catastrophic holography'. *Phys. Rev.* D60 (1999), p. 064018. doi:10.1103/PhysRevD.60.064018. arXiv:hep-th/9902170 [hep-th].

[61] Mirjam Cvetic and Steven S. Gubser. 'Phases of R charged black holes, spinning branes and strongly coupled gauge theories'. *JHEP* 04 (1999), p. 024. doi:10.1088/1126-6708/1999/04/024. arXiv:hep-th/9902195 [hep-th].

[62] S. W. Hawking. 'Particle Creation by Black Holes'. *Commun. Math. Phys.* 43 (1975). [,167(1975)], pp. 199–220. doi:10.1007/BF02345020,10.1007/BF01608497.

[63] Jacob D. Bekenstein. 'Black holes and entropy'. *Phys. Rev.* D7 (1973), pp. 2333–2346. doi:10.1103/PhysRevD.7.2333.

[64] James M. Bardeen, B. Carter and S. W. Hawking. 'The Four laws of black hole mechanics'. *Commun. Math. Phys.* 31 (1973), pp. 161–170. doi:10.1007/BF01645742.

[65] Robert M. Wald. ‘The thermodynamics of black holes’. *Living Rev. Rel.* 4 (2001), p. 6. doi:10.12942/lrr-2001-6. arXiv:gr-qc/9912119 [gr-qc].

[66] Simon F. Ross. ‘Black hole thermodynamics’ (2005). arXiv:hep-th/0502195 [hep-th].

[67] Robert M. Wald. ‘Black hole entropy is the Noether charge’. *Phys. Rev.* D48.8 (1993), R3427–R3431. doi:10.1103/PhysRevD.48.R3427. arXiv:gr-qc/9307038 [gr-qc].

[68] Vivek Iyer and Robert M. Wald. ‘Some properties of Noether charge and a proposal for dynamical black hole entropy’. *Phys. Rev.* D50 (1994), pp. 846–864. doi:10.1103/PhysRevD.50.846. arXiv:gr-qc/9403028 [gr-qc].

[69] Aitor Lewkowycz and Juan Maldacena. ‘Generalized gravitational entropy’. *JHEP* 08 (2013), p. 090. doi:10.1007/JHEP08(2013)090. arXiv:1304.4926 [hep-th].

[70] S. W. Hawking and Don N. Page. ‘Thermodynamics of Black Holes in anti-De Sitter Space’. *Commun. Math. Phys.* 87 (1983), p. 577. doi:10.1007/BF01208266.

[71] Thomas Faulkner, Hong Liu, John McGreevy and David Vegh. ‘Emergent quantum criticality, Fermi surfaces, and AdS(2)’. *Phys.Rev.* D83 (2011), p. 125002. doi:10.1103/PhysRevD.83.125002. arXiv:0907.2694 [hep-th].

[72] Nabil Iqbal, Hong Liu and Mark Mezei. ‘Lectures on holographic non-Fermi liquids and quantum phase transitions’. *Proceedings, Theoretical Advanced Study Institute in Elementary Particle Physics (TASI 2010). String Theory and Its Applications: From meV to the Planck Scale: Boulder, Colorado, USA, June 1-25, 2010.* 2011, pp. 707–816. doi:10.1142/9789814350525_0013. arXiv:1110.3814 [hep-th].

[73] Juan Maldacena and Douglas Stanford. ‘Remarks on the Sachdev-Ye-Kitaev model’. *Phys. Rev.* D94.10 (2016), p. 106002. doi:10.1103/PhysRevD.94.106002. arXiv:1604.07818 [hep-th].

[74] Shamit Kachru, Xiao Liu and Michael Mulligan. ‘Gravity duals of Lifshitz-like fixed points’. *Phys. Rev.* D78 (2008), p. 106005. doi:10.1103/PhysRevD.78.106005. arXiv:0808.1725 [hep-th].

[75] Marika Taylor. ‘Non-relativistic holography’ (2008). arXiv:0812.0530 [hep-th].

[76] Kevin Goldstein, Shamit Kachru, Shiroman Prakash and Sandip P. Trivedi. ‘Holography of Charged Dilaton Black Holes’. *JHEP* 08 (2010), p. 078. doi:10.1007/JHEP08(2010)078. arXiv:0911.3586 [hep-th].

[77] Sean A. Hartnoll, Joseph Polchinski, Eva Silverstein and David Tong. 'Towards strange metallic holography'. *JHEP* 04 (2010), p. 120. `doi:10.1007/JHEP04(2010)120`. `arXiv:0912.1061 [hep-th]`.

[78] Christopher P. Herzog, Mukund Rangamani and Simon F. Ross. 'Heating up Galilean holography'. *JHEP* 11 (2008), p. 080. `doi:10.1088/1126-6708/2008/11/080`. `arXiv:0807.1099 [hep-th]`.

[79] Juan Maldacena, Dario Martelli and Yuji Tachikawa. 'Comments on string theory backgrounds with non-relativistic conformal symmetry'. *JHEP* 10 (2008), p. 072. `doi:10.1088/1126-6708/2008/10/072`. `arXiv:0807.1100 [hep-th]`.

[80] Christos Charmousis, Blaise Gouteraux, Bom Soo Kim, Elias Kiritsis and Rene Meyer. 'Effective Holographic Theories for low-temperature condensed matter systems'. *JHEP* 11 (2010), p. 151. `doi:10.1007/JHEP11(2010)151`. `arXiv:1005.4690 [hep-th]`.

[81] Liza Huijse, Subir Sachdev and Brian Swingle. 'Hidden Fermi surfaces in compressible states of gauge-gravity duality'. *Phys.Rev.* B85 (2012), p. 035121. `doi:10.1103/PhysRevB.85.035121`. `arXiv:1112.0573 [cond-mat.str-el]`.

[82] B. Gouteraux and E. Kiritsis. 'Generalized Holographic Quantum Criticality at Finite Density'. *JHEP* 12 (2011), p. 036. `doi:10.1007/JHEP12(2011)036`. `arXiv:1107.2116 [hep-th]`.

[83] S. Sachdev. *Quantum Phase Transitions*. Cambridge University Press, 2011.

[84] Matthias Troyer and Uwe-Jens Wiese. 'Computational complexity and fundamental limitations to fermionic quantum Monte Carlo simulations'. *Phys. Rev. Lett.* 94 (2005), p. 170201. `doi:10.1103/PhysRevLett.94.170201`. `arXiv:cond-mat/0408370 [cond-mat]`.

[85] Nabil Iqbal and Hong Liu. 'Universality of the hydrodynamic limit in AdS/CFT and the membrane paradigm'. *Phys.Rev.* D79 (2009), p. 025023. `doi:10.1103/PhysRevD.79.025023`. `arXiv:0809.3808 [hep-th]`.

[86] Nabil Iqbal and Hong Liu. 'Real-time response in AdS/CFT with application to spinors'. *Fortsch. Phys.* 57 (2009), pp. 367–384. `doi:10.1002/prop.200900057`. `arXiv:0903.2596 [hep-th]`.

[87] Sean A. Hartnoll. 'Lectures on holographic methods for condensed matter physics'. *Class.Quant.Grav.* 26 (2009), p. 224002. `doi:10.1088/0264-9381/26/22/224002`. `arXiv:0903.3246 [hep-th]`.

[88] Giuseppe Policastro, Dam T. Son and Andrei O. Starinets. 'From AdS / CFT correspondence to hydrodynamics'. *JHEP* 09 (2002), p. 043. doi:10.1088/1126-6708/2002/09/043. arXiv:hep-th/0205052 [hep-th].

[89] Dam T. Son and Andrei O. Starinets. 'Minkowski space correlators in AdS / CFT correspondence: Recipe and applications'. *JHEP* 0209 (2002), p. 042. doi:10.1088/1126-6708/2002/09/042. arXiv:hep-th/0205051 [hep-th].

[90] T. Damour. 'Black Hole Eddy Currents'. *Phys. Rev.* D18 (1978), pp. 3598–3604. doi:10.1103/PhysRevD.18.3598.

[91] Kip S. Thorne, R. H. Price and D. A. Macdonald, eds. *BLACK HOLES: THE MEMBRANE PARADIGM.* 1986.

[92] C. P. Herzog and D. T. Son. 'Schwinger-Keldysh propagators from AdS/CFT correspondence'. *JHEP* 03 (2003), p. 046. doi:10.1088/1126-6708/2003/03/046. arXiv:hep-th/0212072 [hep-th].

[93] Julian S. Schwinger. 'Brownian motion of a quantum oscillator'. *J. Math. Phys.* 2 (1961), pp. 407–432. doi:10.1063/1.1703727.

[94] L. V. Keldysh. 'Diagram technique for nonequilibrium processes'. *Zh. Eksp. Teor. Fiz.* 47 (1964). [Sov. Phys. JETP20,1018(1965)], pp. 1515–1527.

[95] Kostas Skenderis and Balt C. van Rees. 'Real-time gauge/gravity duality'. *Phys. Rev. Lett.* 101 (2008), p. 081601. doi:10.1103/PhysRevLett.101.081601. arXiv:0805.0150 [hep-th].

[96] Kostas Skenderis and Balt C. van Rees. 'Real-time gauge/gravity duality: Prescription, Renormalization and Examples'. *JHEP* 05 (2009), p. 085. doi:10.1088/1126-6708/2009/05/085. arXiv:0812.2909 [hep-th].

[97] Balt C. van Rees. 'Real-time gauge/gravity duality and ingoing boundary conditions'. *Nucl. Phys. Proc. Suppl.* 192-193 (2009), pp. 193–196. doi:10.1016/j.nuclphysbps.2009.07.078. arXiv:0902.4010 [hep-th].

[98] Stephen H. Shenker and Douglas Stanford. 'Black holes and the butterfly effect'. *JHEP* 03 (2014), p. 067. doi:10.1007/JHEP03(2014)067. arXiv:1306.0622 [hep-th].

[99] Stephen H. Shenker and Douglas Stanford. 'Multiple Shocks'. *JHEP* 12 (2014), p. 046. doi:10.1007/JHEP12(2014)046. arXiv:1312.3296 [hep-th].

[100] Daniel A. Roberts, Douglas Stanford and Leonard Susskind. 'Localized shocks'. *JHEP* 03 (2015), p. 051. doi:10.1007/JHEP03(2015)051. arXiv:1409.8180 [hep-th].

[101] Daniel A. Roberts and Douglas Stanford. 'Two-dimensional conformal field theory and the butterfly effect'. *Phys. Rev. Lett.* 115.13 (2015), p. 131603. `doi:10.1103/PhysRevLett.115.131603`. `arXiv:1412.5123 [hep-th]`.

[102] Juan Maldacena, Stephen H. Shenker and Douglas Stanford. 'A bound on chaos'. *JHEP* 08 (2016), p. 106. `doi:10.1007/JHEP08(2016)106`. `arXiv:1503.01409 [hep-th]`.

[103] Mike Blake. 'Universal Diffusion in Incoherent Black Holes'. *Phys. Rev.* D94.8 (2016), p. 086014. `doi:10.1103/PhysRevD.94.086014`. `arXiv:1604.01754 [hep-th]`.

[104] Mike Blake. 'Universal Charge Diffusion and the Butterfly Effect in Holographic Theories'. *Phys. Rev. Lett.* 117.9 (2016), p. 091601. `doi:10.1103/PhysRevLett.117.091601`. `arXiv:1603.08510 [hep-th]`.

[105] Mike Blake and Aristomenis Donos. 'Diffusion and Chaos from near AdS_2 horizons'. *JHEP* 02 (2017), p. 013. `doi:10.1007/JHEP02(2017)013`. `arXiv:1611.09380 [hep-th]`.

[106] Mike Blake, Richard A. Davison and Subir Sachdev. 'Thermal diffusivity and chaos in metals without quasiparticles'. *Phys. Rev.* D96.10 (2017), p. 106008. `doi:10.1103/PhysRevD.96.106008`. `arXiv:1705.07896 [hep-th]`.

[107] Sašo Grozdanov, Koenraad Schalm and Vincenzo Scopelliti. 'Black hole scrambling from hydrodynamics' (2017). `arXiv:1710.00921 [hep-th]`.

[108] Mike Blake, Hyunseok Lee and Hong Liu. 'A quantum hydrodynamical description for scrambling and many-body chaos' (2017). `arXiv:1801.00010 [hep-th]`.

[109] Subir Sachdev. *Quantum Phase Transitions*. 2nd ed. Cambridge University Press, 2011. `doi:10.1017/CBO9780511973765`.

[110] P. M. Chaikin and T. C. Lubensky. *Principles of Condensed Matter Physics*. Cambridge University Press, 2000.

[111] Subir Sachdev. 'The Quantum phases of matter'. *Proceedings, 25th Solvay Conference on Physics: The Theory of the Quantum World: Brussels, Belgium, October 19-25, 2011*. 2012. `arXiv:1203.4565 [hep-th]`.

[112] V. J. Emery and S. A. Kivelson. 'Superconductivity in Bad Metals'. *Phys. Rev. Lett.* 74 (16 Apr. 1995), pp. 3253–3256. `https://link.aps.org/doi/10.1103/PhysRevLett.74.3253doi:10.1103/PhysRevLett.74.3253`.

[113] Richard A. Davison, Koenraad Schalm and Jan Zaanen. 'Holographic duality and the resistivity of strange metals'. *Phys. Rev.* B89.24 (2014), p. 245116. doi:10.1103/PhysRevB.89.245116. arXiv:1311.2451 [hep-th].

[114] O. Gunnarsson, M. Calandra and J.E. Han. 'Colloquium: Saturation of electrical resistivity'. *Rev.Mod.Phys.* 75 (2003), pp. 1085–1099. doi:10.1103/RevModPhys.75.1085. arXiv:cond-mat/0305412 [cond-mat.str-el].

[115] N. E. Hussey, K. Takenaka and H Takagi. 'Universality of the Mott-Ioffe-Regel limit in metals'. *Phil. Mag.* 84 (2004), pp. 2847–2864. arXiv:cond-mat/0404263 [cond-mat.str-el].

[116] Cyril Proust, Kamran Behnia, Romain Bel, Duncan Maude and S. I. Vedeneev. 'Heat transport in $\mathrm{Bi}_{2+x}\mathrm{Sr}_{2-x}\mathrm{CuO}_{6+\delta}$: Departure from the Wiedemann-Franz law in the vicinity of the metal-insulator transition'. *Phys. Rev. B* 72 (21 Dec. 2005), p. 214511. https://link.aps.org/doi/10.1103/PhysRevB.72.214511doi:10.1103/PhysRevB.72.214511.

[117] Raghu Mahajan, Maissam Barkeshli and Sean A. Hartnoll. 'Non-Fermi liquids and the Wiedemann-Franz law'. *Phys. Rev.* B88 (2013), p. 125107. doi:10.1103/PhysRevB.88.125107. arXiv:1304.4249 [cond-mat.str-el].

[118] John Bardeen, L. N. Cooper and J. R. Schrieffer. 'Theory of superconductivity'. *Phys. Rev.* 108 (1957), pp. 1175–1204. doi:10.1103/PhysRev.108.1175.

[119] Subir Sachdev and Bernhard Keimer. 'Quantum Criticality'. *Phys. Today* 64N2 (2011), p. 29. doi:10.1063/1.3554314. arXiv:1102.4628 [cond-mat.str-el].

[120] Jan Zaanen, Yan Liu, Ya-Wen Sun and Koenraad Schalm. *Holographic Duality in Condensed Matter Physics*. Cambridge University Press, 2015. doi:10.1017/CBO9781139942492.

[121] Martin Ammon and Johanna Erdmenger. *Gauge/Gravity Duality: Foundations and Applications*. Cambridge University Press, 2015. doi:10.1017/CBO9780511846373.

[122] Horatiu Nastase. *String Theory Methods for Condensed Matter Physics*. Cambridge University Press, 2017.

[123] Sean A. Hartnoll, Andrew Lucas and Subir Sachdev. 'Holographic quantum matter' (2016). arXiv:1612.07324 [hep-th].

[124] Christopher P. Herzog. 'Lectures on Holographic Superfluidity and Superconductivity'. *J. Phys.* A42 (2009), p. 343001. `doi:10.1088/1751-8113/42/34/343001`. `arXiv:0904.1975 [hep-th]`.

[125] John McGreevy. 'Holographic duality with a view toward many-body physics'. *Adv. High Energy Phys.* 2010 (2010), p. 723105. `doi:10.1155/2010/723105`. `arXiv:0909.0518 [hep-th]`.

[126] Sean A. Hartnoll. 'Horizons, holography and condensed matter' (2011). `arXiv:1106.4324 [hep-th]`.

[127] Gary T. Horowitz. 'Introduction to Holographic Superconductors'. *Lect. Notes Phys.* 828 (2011), pp. 313–347. `doi:10.1007/978-3-642-04864-7_10`. `arXiv:1002.1722 [hep-th]`.

[128] Thomas Faulkner, Nabil Iqbal, Hong Liu, John McGreevy and David Vegh. 'From Black Holes to Strange Metals' (2010). `arXiv:1003.1728 [hep-th]`.

[129] Thomas Faulkner, Nabil Iqbal, Hong Liu, John McGreevy and David Vegh. 'Strange metal transport realized by gauge/gravity duality'. *Science* 329 (2010), pp. 1043–1047. `doi:10.1126/science.1189134`.

[130] J. M. Luttinger. 'Theory of Thermal Transport Coefficients'. *Phys. Rev.* 135 (1964), A1505–A1514. `doi:10.1103/PhysRev.135.A1505`.

[131] D. Forster. *Hydrodynamic Fluctuations, Broken Symmetry, and Correlation Functions*. Perseus Books, 1990, pp. 1–326.

[132] Sean A. Hartnoll, Pavel K. Kovtun, Markus Muller and Subir Sachdev. 'Theory of the Nernst effect near quantum phase transitions in condensed matter, and in dyonic black holes'. *Phys. Rev.* B76 (2007), p. 144502. `doi:10.1103/PhysRevB.76.144502`. `arXiv:0706.3215 [cond-mat.str-el]`.

[133] Sean A. Hartnoll and Christopher P. Herzog. 'Ohm's Law at strong coupling: S duality and the cyclotron resonance'. *Phys. Rev.* D76 (2007), p. 106012. `doi:10.1103/PhysRevD.76.106012`. `arXiv:0706.3228 [hep-th]`.

[134] Richard A. Davison, Blaise Goutéraux and Sean A. Hartnoll. 'Incoherent transport in clean quantum critical metals'. *JHEP* 10 (2015), p. 112. `doi:10.1007/JHEP10(2015)112`. `arXiv:1507.07137 [hep-th]`.

[135] Andrew Lucas. 'Hydrodynamic transport in strongly coupled disordered quantum field theories'. *New J. Phys.* 17.11 (2015), p. 113007. `doi:10.1088/1367-2630/17/11/113007`. `arXiv:1506.02662 [hep-th]`.

[136] L. D. Landau and E. M. Lifshitz. *Fluid Mechanics (Second Edition)*. Pergamon, 1987.

[137] Pavel Kovtun. 'Lectures on hydrodynamic fluctuations in relativistic theories'. *J. Phys.* A45 (2012), p. 473001. doi:10.1088/1751-8113/45/47/473001. arXiv:1205.5040 [hep-th].

[138] Mukund Rangamani. 'Gravity and Hydrodynamics: Lectures on the fluid-gravity correspondence'. *Class. Quant. Grav.* 26 (2009), p. 224003. doi:10.1088/0264-9381/26/22/224003. arXiv:0905.4352 [hep-th].

[139] Paul Romatschke. 'New Developments in Relativistic Viscous Hydrodynamics'. *Int. J. Mod. Phys.* E19 (2010), pp. 1–53. doi:10.1142/S0218301310014613. arXiv:0902.3663 [hep-ph].

[140] Paul Romatschke and Ulrike Romatschke. 'Relativistic Fluid Dynamics In and Out of Equilibrium – Ten Years of Progress in Theory and Numerical Simulations of Nuclear Collisions' (2017). arXiv:1712.05815 [nucl-th].

[141] D. T. Son. 'Hydrodynamics of relativistic systems with broken continuous symmetries'. *Int. J. Mod. Phys.* A16S1C (2001), pp. 1284–1286. doi:10.1142/S0217751X01009545. arXiv:hep-ph/0011246 [hep-ph].

[142] L. P. Kadanoff and P. C. Martin. 'Hydrodynamic equations and correlation functions'. *Annals of Physics* 24 (Oct. 1963), pp. 419–469. doi:10.1016/0003-4916(63)90078-2.

[143] Sayantani Bhattacharyya, Veronika E Hubeny, Shiraz Minwalla and Mukund Rangamani. 'Nonlinear Fluid Dynamics from Gravity'. *JHEP* 0802 (2008), p. 045. doi:10.1088/1126-6708/2008/02/045. arXiv:0712.2456 [hep-th].

[144] Sayantani Bhattacharyya, Subhaneil Lahiri, R. Loganayagam and Shiraz Minwalla. 'Large rotating AdS black holes from fluid mechanics'. *JHEP* 09 (2008), p. 054. doi:10.1088/1126-6708/2008/09/054. arXiv:0708.1770 [hep-th].

[145] Sayantani Bhattacharyya, R. Loganayagam, Ipsita Mandal, Shiraz Minwalla and Ankit Sharma. 'Conformal Nonlinear Fluid Dynamics from Gravity in Arbitrary Dimensions'. *JHEP* 12 (2008), p. 116. doi:10.1088/1126-6708/2008/12/116. arXiv:0809.4272 [hep-th].

[146] Veronika E. Hubeny, Shiraz Minwalla and Mukund Rangamani. 'The fluid/gravity correspondence'. *Black holes in higher dimensions.* [,817(2011)]. 2012, pp. 348–383. http://inspirehep.net/record/920982/files/arXiv:1107.5780.pdf. arXiv:1107.5780 [hep-th].

[147] Veronika E. Hubeny. 'Fluid Dynamics from Gravity'. *From Gravity to Thermal Gauge Theories: The AdS/CFT Correspondence: The AdS/CFT Correspondence*. Ed. by Eleftherios Papantonopoulos. Berlin, Heidelberg: Springer Berlin Heidelberg, 2011, pp. 183–203. `https://doi.org/10.1007/978-3-642-04864-7_6doi:10.1007/978-3-642-04864-7_6`.

[148] Michael Haack and Amos Yarom. 'Nonlinear viscous hydrodynamics in various dimensions using AdS/CFT'. *JHEP* 10 (2008), p. 063. `doi:10.1088/1126-6708/2008/10/063`. `arXiv:0806.4602 [hep-th]`.

[149] Rudolf Baier, Paul Romatschke, Dam Thanh Son, Andrei O. Starinets and Mikhail A. Stephanov. 'Relativistic viscous hydrodynamics, conformal invariance, and holography'. *JHEP* 04 (2008), p. 100. `doi:10.1088/1126-6708/2008/04/100`. `arXiv:0712.2451 [hep-th]`.

[150] Julian Sonner and Benjamin Withers. 'A gravity derivation of the Tisza-Landau Model in AdS/CFT'. *Phys. Rev.* D82 (2010), p. 026001. `doi:10.1103/PhysRevD.82.026001`. `arXiv:1004.2707 [hep-th]`.

[151] Christopher P. Herzog, Nir Lisker, Piotr Surowka and Amos Yarom. 'Transport in holographic superfluids'. *JHEP* 08 (2011), p. 052. `doi:10.1007/JHEP08(2011)052`. `arXiv:1101.3330 [hep-th]`.

[152] Jyotirmoy Bhattacharya, Sayantani Bhattacharyya, Shiraz Minwalla and Amos Yarom. 'A Theory of first order dissipative superfluid dynamics'. *JHEP* 05 (2014), p. 147. `doi:10.1007/JHEP05(2014)147`. `arXiv:1105.3733 [hep-th]`.

[153] Veronika E. Hubeny, Mukund Rangamani, Shiraz Minwalla and Mark Van Raamsdonk. 'The fluid-gravity correspondence: The membrane at the end of the universe'. *Int. J. Mod. Phys.* D17 (2009), pp. 2571–2576. `doi:10.1142/S0218271808014084`.

[154] Veronika E. Hubeny. 'The Fluid/Gravity Correspondence: a new perspective on the Membrane Paradigm'. *Class. Quant. Grav.* 28 (2011), p. 114007. `doi:10.1088/0264-9381/28/11/114007`. `arXiv:1011.4948 [gr-qc]`.

[155] Andrei O. Starinets. 'Quasinormal spectrum and the black hole membrane paradigm'. *Phys. Lett.* B670 (2009), pp. 442–445. `doi:10.1016/j.physletb.2008.11.028`. `arXiv:0806.3797 [hep-th]`.

[156] Sayantani Bhattacharyya, Shiraz Minwalla and Spenta R. Wadia. 'The Incompressible Non-Relativistic Navier-Stokes Equation from Gravity'. *JHEP* 0908 (2009), p. 059. `doi:10.1088/1126-6708/2009/08/059`. `arXiv:0810.1545 [hep-th]`.

[157] Irene Bredberg, Cynthia Keeler, Vyacheslav Lysov and Andrew Strominger. 'Wilsonian Approach to Fluid/Gravity Duality'. *JHEP* 03 (2011), p. 141. `doi:10.1007/JHEP03(2011)141`. `arXiv:1006.1902 [hep-th]`.

[158] Christopher Eling, Itzhak Fouxon and Yaron Oz. 'The Incompressible Navier-Stokes Equations From Membrane Dynamics'. *Phys. Lett.* B680 (2009), pp. 496–499. `doi:10.1016/j.physletb.2009.09.028`. `arXiv:0905.3638 [hep-th]`.

[159] Christopher Eling and Yaron Oz. 'Relativistic CFT Hydrodynamics from the Membrane Paradigm'. *JHEP* 02 (2010), p. 069. `doi:10.1007/JHEP02(2010)069`. `arXiv:0906.4999 [hep-th]`.

[160] Irene Bredberg, Cynthia Keeler, Vyacheslav Lysov and Andrew Strominger. 'From Navier-Stokes To Einstein'. *JHEP* 1207 (2012), p. 146. `doi:10.1007/JHEP07(2012)146`. `arXiv:1101.2451 [hep-th]`.

[161] Michael Crossley, Paolo Glorioso and Hong Liu. 'Effective field theory of dissipative fluids'. *JHEP* 09 (2017), p. 095. `doi:10.1007/JHEP09(2017)095`. `arXiv:1511.03646 [hep-th]`.

[162] Felix M. Haehl, R. Loganayagam and Mukund Rangamani. 'The Fluid Manifesto: Emergent symmetries, hydrodynamics, and black holes'. *JHEP* 01 (2016), p. 184. `doi:10.1007/JHEP01(2016)184`. `arXiv:1510.02494 [hep-th]`.

[163] Felix M. Haehl, R. Loganayagam and M. Rangamani. 'Topological sigma models & dissipative hydrodynamics'. *JHEP* 04 (2016), p. 039. `doi:10.1007/JHEP04(2016)039`. `arXiv:1511.07809 [hep-th]`.

[164] Felix M. Haehl, R. Loganayagam and Mukund Rangamani. 'Two roads to hydrodynamic effective actions: a comparison' (2017). `arXiv:1701.07896 [hep-th]`.

[165] Kristan Jensen, Matthias Kaminski, Pavel Kovtun, Rene Meyer, Adam Ritz and Amos Yarom. 'Parity-Violating Hydrodynamics in 2+1 Dimensions'. *JHEP* 05 (2012), p. 102. `doi:10.1007/JHEP05(2012)102`. `arXiv:1112.4498 [hep-th]`.

[166] Felix M. Haehl, R. Loganayagam and Mukund Rangamani. 'The eightfold way to dissipation'. *Phys. Rev. Lett.* 114 (2015), p. 201601. `doi:10.1103/PhysRevLett.114.201601`. `arXiv:1412.1090 [hep-th]`.

[167] Sayantani Bhattacharyya, Veronika E. Hubeny, R. Loganayagam, Gautam Mandal, Shiraz Minwalla, Takeshi Morita, Mukund Rangamani and Harvey S. Reall. 'Local Fluid Dynamical Entropy from Gravity'. *JHEP* 06 (2008), p. 055. doi:10.1088/1126-6708/2008/06/055. arXiv:0803.2526 [hep-th].

[168] Paolo Glorioso and Hong Liu. 'The second law of thermodynamics from symmetry and unitarity' (2016). arXiv:1612.07705 [hep-th].

[169] Felix M. Haehl, R. Loganayagam and Mukund Rangamani. 'An Inflow Mechanism for Hydrodynamic Entropy' (2018). arXiv:1803.08490 [hep-th].

[170] Kristan Jensen, Raja Marjieh, Natalia Pinzani-Fokeeva and Amos Yarom. 'An entropy current in superspace' (2018). arXiv:1803.07070 [hep-th].

[171] Emanuele Berti, Vitor Cardoso and Andrei O. Starinets. 'Quasinormal modes of black holes and black branes'. *Class. Quant. Grav.* 26 (2009), p. 163001. doi:10.1088/0264-9381/26/16/163001. arXiv:0905.2975 [gr-qc].

[172] R. A. Konoplya and A. Zhidenko. 'Quasinormal modes of black holes: From astrophysics to string theory'. *Rev. Mod. Phys.* 83 (2011), pp. 793–836. doi:10.1103/RevModPhys.83.793. arXiv:1102.4014 [gr-qc].

[173] Gary T. Horowitz and Veronika E. Hubeny. 'Quasinormal modes of AdS black holes and the approach to thermal equilibrium'. *Phys. Rev.* D62 (2000), p. 024027. doi:10.1103/PhysRevD.62.024027. arXiv:hep-th/9909056 [hep-th].

[174] Ulf H. Danielsson, Esko Keski-Vakkuri and Martin Kruczenski. 'Black hole formation in AdS and thermalization on the boundary'. *JHEP* 02 (2000), p. 039. doi:10.1088/1126-6708/2000/02/039. arXiv:hep-th/9912209 [hep-th].

[175] Danny Birmingham, Ivo Sachs and Sergey N. Solodukhin. 'Conformal field theory interpretation of black hole quasinormal modes'. *Phys. Rev. Lett.* 88 (2002), p. 151301. doi:10.1103/PhysRevLett.88.151301. arXiv:hep-th/0112055 [hep-th].

[176] Pavel K. Kovtun and Andrei O. Starinets. 'Quasinormal modes and holography'. *Phys. Rev.* D72 (2005), p. 086009. doi:10.1103/PhysRevD.72.086009. arXiv:hep-th/0506184 [hep-th].

[177] Jose Natario and Ricardo Schiappa. 'On the classification of asymptotic quasinormal frequencies for d-dimensional black holes and quantum gravity'. *Adv. Theor. Math. Phys.* 8.6 (2004), pp. 1001–1131. doi:10.4310/ATMP.2004.v8.n6.a4. arXiv:hep-th/0411267 [hep-th].

[178] Guido Festuccia and Hong Liu. 'A Bohr-Sommerfeld quantization formula for quasinormal frequencies of AdS black holes'. *Adv. Sci. Lett.* 2 (2009), pp. 221–235. doi:10.1166/asl.2009.1029. arXiv:0811.1033 [gr-qc].

[179] Andrei O. Starinets. 'Quasinormal modes of near extremal black branes'. *Phys. Rev.* D66 (2002), p. 124013. doi:10.1103/PhysRevD.66.124013. arXiv:hep-th/0207133 [hep-th].

[180] Alvaro Nunez and Andrei O. Starinets. 'AdS / CFT correspondence, quasinormal modes, and thermal correlators in N=4 SYM'. *Phys. Rev.* D67 (2003), p. 124013. doi:10.1103/PhysRevD.67.124013. arXiv:hep-th/0302026 [hep-th].

[181] William Witczak-Krempa and Subir Sachdev. 'The quasi-normal modes of quantum criticality'. *Phys. Rev.* B86 (2012), p. 235115. doi:10.1103/PhysRevB.86.235115. arXiv:1210.4166 [cond-mat.str-el].

[182] Giuseppe Policastro, Dam T. Son and Andrei O. Starinets. 'From AdS / CFT correspondence to hydrodynamics. 2. Sound waves'. *JHEP* 12 (2002), p. 054. doi:10.1088/1126-6708/2002/12/054. arXiv:hep-th/0210220 [hep-th].

[183] Christopher P. Herzog. 'The Hydrodynamics of M theory'. *JHEP* 12 (2002), p. 026. doi:10.1088/1126-6708/2002/12/026. arXiv:hep-th/0210126 [hep-th].

[184] Christopher P. Herzog. 'The Sound of M theory'. *Phys. Rev.* D68 (2003), p. 024013. doi:10.1103/PhysRevD.68.024013. arXiv:hep-th/0302086 [hep-th].

[185] Pavel Kovtun, Dam T. Son and Andrei O. Starinets. 'Holography and hydrodynamics: Diffusion on stretched horizons'. *JHEP* 10 (2003), p. 064. doi:10.1088/1126-6708/2003/10/064. arXiv:hep-th/0309213 [hep-th].

[186] Dam T. Son and Andrei O. Starinets. 'Hydrodynamics of r-charged black holes'. *JHEP* 03 (2006), p. 052. doi:10.1088/1126-6708/2006/03/052. arXiv:hep-th/0601157 [hep-th].

[187] Kengo Maeda, Makoto Natsuume and Takashi Okamura. 'Viscosity of gauge theory plasma with a chemical potential from AdS/CFT'. *Phys. Rev.* D73 (2006), p. 066013. doi:10.1103/PhysRevD.73.066013. arXiv:hep-th/0602010 [hep-th].

[188] Robert C. Myers, Miguel F. Paulos and Aninda Sinha. 'Holographic Hydrodynamics with a Chemical Potential'. *JHEP* 06 (2009), p. 006. doi:10.1088/1126-6708/2009/06/006. arXiv:0903.2834 [hep-th].

[189] Johanna Erdmenger, Michael Haack, Matthias Kaminski and Amos Yarom. 'Fluid dynamics of R-charged black holes'. *JHEP* 01 (2009), p. 055. `doi:10.1088/1126-6708/2009/01/055`. `arXiv:0809.2488 [hep-th]`.

[190] Ingmar Kanitscheider and Kostas Skenderis. 'Universal hydrodynamics of non-conformal branes'. *JHEP* 04 (2009), p. 062. `doi:10.1088/1126-6708/2009/04/062`. `arXiv:0901.1487 [hep-th]`.

[191] Mohammad Edalati, Juan I. Jottar and Robert G. Leigh. 'Transport Coefficients at Zero Temperature from Extremal Black Holes'. *JHEP* 01 (2010), p. 018. `doi:10.1007/JHEP01(2010)018`. `arXiv:0910.0645 [hep-th]`.

[192] Mohammad Edalati, Juan I. Jottar and Robert G. Leigh. 'Shear Modes, Criticality and Extremal Black Holes'. *JHEP* 04 (2010), p. 075. `doi:10.1007/JHEP04(2010)075`. `arXiv:1001.0779 [hep-th]`.

[193] Irene Amado, Carlos Hoyos-Badajoz, Karl Landsteiner and Sergio Montero. 'Hydrodynamics and beyond in the strongly coupled N=4 plasma'. *JHEP* 07 (2008), p. 133. `doi:10.1088/1126-6708/2008/07/133`. `arXiv:0805.2570 [hep-th]`.

[194] Mohammad Edalati, Juan I. Jottar and Robert G. Leigh. 'Holography and the sound of criticality'. *JHEP* 10 (2010), p. 058. `doi:10.1007/JHEP10(2010)058`. `arXiv:1005.4075 [hep-th]`.

[195] Andrei O. Starinets. 'Transport coefficients of strongly coupled gauge theories: Insights from string theory'. *Eur. Phys. J.* A29 (2006), pp. 77–81. `doi:10.1140/epja/i2005-10301-2`. `arXiv:nucl-th/0511073 [nucl-th]`.

[196] Dam T. Son and Andrei O. Starinets. 'Viscosity, Black Holes, and Quantum Field Theory'. *Ann. Rev. Nucl. Part. Sci.* 57 (2007), pp. 95–118. `doi:10.1146/annurev.nucl.57.090506.123120`. `arXiv:0704.0240 [hep-th]`.

[197] G. Policastro, Dan T. Son and Andrei O. Starinets. 'The Shear viscosity of strongly coupled N=4 supersymmetric Yang-Mills plasma'. *Phys. Rev. Lett.* 87 (2001), p. 081601. `doi:10.1103/PhysRevLett.87.081601`. `arXiv:hep-th/0104066 [hep-th]`.

[198] P. Kovtun, Dan T. Son and Andrei O. Starinets. 'Viscosity in strongly interacting quantum field theories from black hole physics'. *Phys. Rev. Lett.* 94 (2005), p. 111601. `doi:10.1103/PhysRevLett.94.111601`. `arXiv:hep-th/0405231 [hep-th]`.

[199] Zhong-Ying Fan. 'Note on the Noether charge and holographic transports'. *Phys. Rev.* D97.6 (2018), p. 066013. `doi:10.1103/PhysRevD.97.066013`. `arXiv:1801.07870 [hep-th]`.

[200] Jorge Casalderrey-Solana, Hong Liu, David Mateos, Krishna Rajagopal and Urs Achim Wiedemann. 'Gauge/String Duality, Hot QCD and Heavy Ion Collisions' (2011). doi:10.1017/CBO9781139136747. arXiv:1101.0618 [hep-th].

[201] Steven S. Gubser and Andreas Karch. 'From gauge-string duality to strong interactions: A Pedestrian's Guide'. *Ann. Rev. Nucl. Part. Sci.* 59 (2009), pp. 145–168. doi:10.1146/annurev.nucl.010909.083602. arXiv:0901.0935 [hep-th].

[202] Alex Buchel and James T. Liu. 'Universality of the shear viscosity in supergravity'. *Phys. Rev. Lett.* 93 (2004), p. 090602. doi:10.1103/PhysRevLett.93.090602. arXiv:hep-th/0311175 [hep-th].

[203] Alex Buchel, James T. Liu and Andrei O. Starinets. 'Coupling constant dependence of the shear viscosity in N=4 supersymmetric Yang-Mills theory'. *Nucl. Phys.* B707 (2005), pp. 56–68. doi:10.1016/j.nuclphysb.2004.11.055. arXiv:hep-th/0406264 [hep-th].

[204] Aninda Sinha and Robert C. Myers. 'The Viscosity bound in string theory'. *Nucl. Phys.* A830 (2009), pp. 295C–298C. doi:10.1016/j.nuclphysa.2009.09.028. arXiv:0907.4798 [hep-th].

[205] Mauro Brigante, Hong Liu, Robert C. Myers, Stephen Shenker and Sho Yaida. 'Viscosity Bound Violation in Higher Derivative Gravity'. *Phys. Rev.* D77 (2008), p. 126006. doi:10.1103/PhysRevD.77.126006. arXiv:0712.0805 [hep-th].

[206] Mauro Brigante, Hong Liu, Robert C. Myers, Stephen Shenker and Sho Yaida. 'The Viscosity Bound and Causality Violation'. *Phys. Rev. Lett.* 100 (2008), p. 191601. doi:10.1103/PhysRevLett.100.191601. arXiv:0802.3318 [hep-th].

[207] Yevgeny Kats and Pavel Petrov. 'Effect of curvature squared corrections in AdS on the viscosity of the dual gauge theory'. *JHEP* 01 (2009), p. 044. doi:10.1088/1126-6708/2009/01/044. arXiv:0712.0743 [hep-th].

[208] Johanna Erdmenger, Patrick Kerner and Hansjorg Zeller. 'Non-universal shear viscosity from Einstein gravity'. *Phys. Lett.* B699 (2011), pp. 301–304. doi:10.1016/j.physletb.2011.04.009. arXiv:1011.5912 [hep-th].

[209] Makoto Natsuume and Masahiro Ohta. 'The Shear viscosity of holographic superfluids'. *Prog. Theor. Phys.* 124 (2010), pp. 931–951. doi:10.1143/PTP.124.931. arXiv:1008.4142 [hep-th].

[210] Sean A. Hartnoll, David M. Ramirez and Jorge E. Santos. 'Entropy production, viscosity bounds and bumpy black holes'. *JHEP* 03 (2016), p. 170. `doi:10.1007/JHEP03(2016)170`. `arXiv:1601.02757 [hep-th]`.

[211] Sera Cremonini. 'The Shear Viscosity to Entropy Ratio: A Status Report'. *Mod. Phys. Lett.* B25 (2011), pp. 1867–1888. `doi:10.1142/S0217984911027315`. `arXiv:1108.0677 [hep-th]`.

[212] G. Gruner. 'The dynamics of charge-density waves'. *Rev. Mod. Phys.* 60 (1988), pp. 1129–1181. `doi:10.1103/RevModPhys.60.1129`.

[213] G. Gruner. 'The dynamics of spin-density waves'. *Rev. Mod. Phys.* 66 (1994), pp. 1–24. `doi:10.1103/RevModPhys.66.1`.

[214] Matthew Headrick, Sam Kitchen and Toby Wiseman. 'A New approach to static numerical relativity, and its application to Kaluza-Klein black holes'. *Class. Quant. Grav.* 27 (2010), p. 035002. `doi:10.1088/0264-9381/27/3/035002`. `arXiv:0905.1822 [gr-qc]`.

[215] Óscar J. C. Dias, Jorge E. Santos and Benson Way. 'Numerical Methods for Finding Stationary Gravitational Solutions'. *Class. Quant. Grav.* 33.13 (2016), p. 133001. `doi:10.1088/0264-9381/33/13/133001`. `arXiv:1510.02804 [hep-th]`.

[216] Gary T. Horowitz, Jorge E. Santos and David Tong. 'Optical Conductivity with Holographic Lattices'. *JHEP* 07 (2012), p. 168. `doi:10.1007/JHEP07(2012)168`. `arXiv:1204.0519 [hep-th]`.

[217] Gary T. Horowitz, Jorge E. Santos and David Tong. 'Further Evidence for Lattice-Induced Scaling'. *JHEP* 11 (2012), p. 102. `doi:10.1007/JHEP11(2012)102`. `arXiv:1209.1098 [hep-th]`.

[218] Gary T. Horowitz and Jorge E. Santos. 'General Relativity and the Cuprates'. *JHEP* 06 (2013), p. 087. `doi:10.1007/JHEP06(2013)087`. `arXiv:1302.6586 [hep-th]`.

[219] Koushik Balasubramanian and Christopher P. Herzog. 'Losing Forward Momentum Holographically'. *Class. Quant. Grav.* 31 (2014), p. 125010. `doi:10.1088/0264-9381/31/12/125010`. `arXiv:1312.4953 [hep-th]`.

[220] Yi Ling, Chao Niu, Jian-Pin Wu and Zhuo-Yu Xian. 'Holographic Lattice in Einstein-Maxwell-Dilaton Gravity'. *JHEP* 11 (2013), p. 006. `doi:10.1007/JHEP11(2013)006`. `arXiv:1309.4580 [hep-th]`.

[221] Aristomenis Donos and Sean A. Hartnoll. 'Interaction-driven localization in holography'. *Nature Phys.* 9 (2013), pp. 649–655. `doi:10.1038/nphys2701`. `arXiv:1212.2998 [hep-th]`.

[222] Aristomenis Donos and Jerome P. Gauntlett. 'Novel metals and insulators from holography'. *JHEP* 06 (2014), p. 007. `doi:10.1007/JHEP06(2014)007`. `arXiv:1401.5077 [hep-th]`.

[223] Aristomenis Donos, Blaise Goutéraux and Elias Kiritsis. 'Holographic Metals and Insulators with Helical Symmetry'. *JHEP* 09 (2014), p. 038. `doi:10.1007/JHEP09(2014)038`. `arXiv:1406.6351 [hep-th]`.

[224] B. Goutéraux. 'Charge transport in holography with momentum dissipation'. *JHEP* 04 (2014), p. 181. `doi:10.1007/JHEP04(2014)181`. `arXiv:1401.5436 [hep-th]`.

[225] Keun-Young Kim, Kyung Kiu Kim, Yunseok Seo and Sang-Jin Sin. 'Coherent/incoherent metal transition in a holographic model'. *JHEP* 12 (2014), p. 170. `doi:10.1007/JHEP12(2014)170`. `arXiv:1409.8346 [hep-th]`.

[226] Mukund Rangamani, Moshe Rozali and Darren Smyth. 'Spatial Modulation and Conductivities in Effective Holographic Theories'. *JHEP* 07 (2015), p. 024. `doi:10.1007/JHEP07(2015)024`. `arXiv:1505.05171 [hep-th]`.

[227] Blaise Goutéraux, Elias Kiritsis and Wei-Jia Li. 'Effective holographic theories of momentum relaxation and violation of conductivity bound'. *JHEP* 04 (2016), p. 122. `doi:10.1007/JHEP04(2016)122`. `arXiv:1602.01067 [hep-th]`.

[228] Matteo Baggioli and Oriol Pujolas. 'On holographic disorder-driven metal-insulator transitions'. *JHEP* 01 (2017), p. 040. `doi:10.1007/JHEP01(2017)040`. `arXiv:1601.07897 [hep-th]`.

[229] Sean A. Hartnoll and Christopher P. Herzog. 'Impure AdS/CFT correspondence'. *Phys. Rev.* D77 (2008), p. 106009. `doi:10.1103/PhysRevD.77.106009`. `arXiv:0801.1693 [hep-th]`.

[230] Sean A. Hartnoll and Diego M. Hofman. 'Locally Critical Resistivities from Umklapp Scattering'. *Phys.Rev.Lett.* 108 (2012), p. 241601. `doi:10.1103/PhysRevLett.108.241601`. `arXiv:1201.3917 [hep-th]`.

[231] Sean A. Hartnoll. 'Theory of universal incoherent metallic transport'. *Nature Phys.* 11 (2015), p. 54. `doi:10.1038/nphys3174`. `arXiv:1405.3651 [cond-mat.str-el]`.

[232] Andrew Lucas. 'Conductivity of a strange metal: from holography to memory functions'. *JHEP* 03 (2015), p. 071. `doi:10.1007/JHEP03(2015)071`. `arXiv:1501.05656 [hep-th]`.

[233] Andrew Lucas, Jesse Crossno, Kin Chung Fong, Philip Kim and Subir Sachdev. 'Transport in inhomogeneous quantum critical fluids and in the Dirac fluid in graphene'. *Phys. Rev.* B93.7 (2016), p. 075426. doi:10.1103/PhysRevB.93.075426. arXiv:1510.01738 [cond-mat.str-el].

[234] Mike Blake. 'Momentum relaxation from the fluid/gravity correspondence'. *JHEP* 09 (2015), p. 010. doi:10.1007/JHEP09(2015)010. arXiv:1505.06992 [hep-th].

[235] Andrew Lucas and Subir Sachdev. 'Memory matrix theory of magnetotransport in strange metals'. *Phys. Rev.* B91.19 (2015), p. 195122. doi:10.1103/PhysRevB.91.195122. arXiv:1502.04704 [cond-mat.str-el].

[236] Mike Blake, Aristomenis Donos and Nakarin Lohitsiri. 'Magnetothermoelectric Response from Holography'. *JHEP* 08 (2015), p. 124. doi:10.1007/JHEP08(2015)124. arXiv:1502.03789 [hep-th].

[237] Richard A. Davison, Luca V. Delacrétaz, Blaise Goutéraux and Sean A. Hartnoll. 'Hydrodynamic theory of quantum fluctuating superconductivity'. *Phys. Rev.* B94.5 (2016). [Erratum: Phys. Rev.B96,no.5,059902(2017)], p. 054502. doi:10.1103/PhysRevB.96.059902,10.1103/PhysRevB.94.054502. arXiv:1602.08171 [cond-mat.supr-con].

[238] Richard A. Davison and Blaise Goutéraux. 'Momentum dissipation and effective theories of coherent and incoherent transport'. *JHEP* 01 (2015), p. 039. doi:10.1007/JHEP01(2015)039. arXiv:1411.1062 [hep-th].

[239] Andrew Lucas, Subir Sachdev and Koenraad Schalm. 'Scale-invariant hyperscaling-violating holographic theories and the resistivity of strange metals with random-field disorder'. *Phys. Rev.* D89.6 (2014), p. 066018. doi:10.1103/PhysRevD.89.066018. arXiv:1401.7993 [hep-th].

[240] Sean A. Hartnoll and Jorge E. Santos. 'Disordered horizons: Holography of randomly disordered fixed points'. *Phys. Rev. Lett.* 112 (2014), p. 231601. doi:10.1103/PhysRevLett.112.231601. arXiv:1402.0872 [hep-th].

[241] Andrew Lucas and Subir Sachdev. 'Conductivity of weakly disordered strange metals: from conformal to hyperscaling-violating regimes'. *Nucl. Phys.* B892 (2015), pp. 239–268. doi:10.1016/j.nuclphysb.2015.01.017. arXiv:1411.3331 [hep-th].

[242] Sean A. Hartnoll, David M. Ramirez and Jorge E. Santos. 'Emergent scale invariance of disordered horizons'. *JHEP* 09 (2015), p. 160. doi:10.1007/JHEP09(2015)160. arXiv:1504.03324 [hep-th].

[243] Ofer Aharony, Zohar Komargodski and Shimon Yankielowicz. 'Disorder in Large-N Theories'. *JHEP* 04 (2016), p. 013. `doi:10.1007/JHEP04(2016)013`. `arXiv:1509.02547 [hep-th]`.

[244] Vladimir Narovlansky and Ofer Aharony. 'The renormalization group in quantum quenched disorder' (2018). `arXiv:1803.08529 [cond-mat.str-el]`.

[245] Ofer Aharony and Vladimir Narovlansky. 'The renormalization group flow in field theories with quenched disorder' (2018). `arXiv:1803.08534 [hep-th]`.

[246] Sean A. Hartnoll, Christopher P. Herzog and Gary T. Horowitz. 'Building a Holographic Superconductor'. *Phys. Rev. Lett.* 101 (2008), p. 031601. `doi:10.1103/PhysRevLett.101.031601`. `arXiv:0803.3295 [hep-th]`.

[247] Sean A. Hartnoll, Christopher P. Herzog and Gary T. Horowitz. 'Holographic Superconductors'. *JHEP* 12 (2008), p. 015. `doi:10.1088/1126-6708/2008/12/015`. `arXiv:0810.1563 [hep-th]`.

[248] Steven S. Gubser. 'Breaking an Abelian gauge symmetry near a black hole horizon'. *Phys. Rev.* D78 (2008), p. 065034. `doi:10.1103/PhysRevD.78.065034`. `arXiv:0801.2977 [hep-th]`.

[249] Steven S. Gubser and Silviu S. Pufu. 'The Gravity dual of a p-wave superconductor'. *JHEP* 11 (2008), p. 033. `doi:10.1088/1126-6708/2008/11/033`. `arXiv:0805.2960 [hep-th]`.

[250] Steven S. Gubser, Silviu S. Pufu and Fabio D. Rocha. 'Quantum critical superconductors in string theory and M-theory'. *Phys. Lett.* B683 (2010), pp. 201–204. `doi:10.1016/j.physletb.2009.12.017`. `arXiv:0908.0011 [hep-th]`.

[251] Sebastian Franco, Antonio Garcia-Garcia and Diego Rodriguez-Gomez. 'A General class of holographic superconductors'. *JHEP* 04 (2010), p. 092. `doi:10.1007/JHEP04(2010)092`. `arXiv:0906.1214 [hep-th]`.

[252] Martin Ammon, Johanna Erdmenger, Viviane Grass, Patrick Kerner and Andy O'Bannon. 'On Holographic p-wave Superfluids with Back-reaction'. *Phys. Lett.* B686 (2010), pp. 192–198. `doi:10.1016/j.physletb.2010.02.021`. `arXiv:0912.3515 [hep-th]`.

[253] Christopher P. Herzog. 'An Analytic Holographic Superconductor'. *Phys. Rev.* D81 (2010), p. 126009. `doi:10.1103/PhysRevD.81.126009`. `arXiv:1003.3278 [hep-th]`.

[254] Jian-Huang She, Bas J. Overbosch, Ya-Wen Sun, Yan Liu, Koenraad Schalm, John A. Mydosh and Jan Zaanen. 'Observing the origin of superconductivity in quantum critical metals'. *Phys. Rev.* B84 (2011), p. 144527. `doi:10.1103/PhysRevB.84.144527`. `arXiv:1105.5377 [cond-mat.str-el]`.

[255] Nabil Iqbal, Hong Liu and Mark Mezei. 'Quantum phase transitions in semilocal quantum liquids'. *Phys. Rev.* D91.2 (2015), p. 025024. `doi:10.1103/PhysRevD.91.025024`. `arXiv:1108.0425 [hep-th]`.

[256] Aristomenis Donos, Jerome P. Gauntlett and Christiana Pantelidou. 'Spatially modulated instabilities of magnetic black branes'. *JHEP* 01 (2012), p. 061. `doi:10.1007/JHEP01(2012)061`. `arXiv:1109.0471 [hep-th]`.

[257] Steven S. Gubser and Indrajit Mitra. 'Instability of charged black holes in Anti-de Sitter space' (2000). `arXiv:hep-th/0009126 [hep-th]`.

[258] Steven S. Gubser and Indrajit Mitra. 'The Evolution of Unstable Black Holes in Anti-de Sitter Space'. *JHEP* 08 (2001), p. 018. `arXiv:hep-th/0011127`.

[259] Shin Nakamura, Hirosi Ooguri and Chang-Soon Park. 'Gravity Dual of Spatially Modulated Phase'. *Phys. Rev.* D81 (2010), p. 044018. `doi:10.1103/PhysRevD.81.044018`. `arXiv:0911.0679 [hep-th]`.

[260] Hirosi Ooguri and Chang-Soon Park. 'Holographic End-Point of Spatially Modulated Phase Transition'. *Phys. Rev.* D82 (2010), p. 126001. `doi:10.1103/PhysRevD.82.126001`. `arXiv:1007.3737 [hep-th]`.

[261] Aristomenis Donos and Jerome P. Gauntlett. 'Black holes dual to helical current phases'. *Phys. Rev.* D86 (2012), p. 064010. `doi:10.1103/PhysRevD.86.064010`. `arXiv:1204.1734 [hep-th]`.

[262] Aristomenis Donos and Jerome P. Gauntlett. 'Holographic striped phases'. *JHEP* 08 (2011), p. 140. `doi:10.1007/JHEP08(2011)140`. `arXiv:1106.2004 [hep-th]`.

[263] Oren Bergman, Niko Jokela, Gilad Lifschytz and Matthew Lippert. 'Striped instability of a holographic Fermi-like liquid'. *JHEP* 10 (2011), p. 034. `doi:10.1007/JHEP10(2011)034`. `arXiv:1106.3883 [hep-th]`.

[264] Aristomenis Donos and Jerome P. Gauntlett. 'Holographic charge density waves'. *Phys. Rev.* D87.12 (2013), p. 126008. `doi:10.1103/PhysRevD.87.126008`. `arXiv:1303.4398 [hep-th]`.

[265] Moshe Rozali, Darren Smyth, Evgeny Sorkin and Jared B. Stang. 'Holographic Stripes'. *Phys. Rev. Lett.* 110.20 (2013), p. 201603. `doi:10.1103/PhysRevLett.110.201603`. `arXiv:1211.5600 [hep-th]`.

[266] Aristomenis Donos. 'Striped phases from holography'. *JHEP* 05 (2013), p. 059. `doi:10.1007/JHEP05(2013)059`. `arXiv:1303.7211 [hep-th]`.

[267] Benjamin Withers. 'Black branes dual to striped phases'. *Class. Quant. Grav.* 30 (2013), p. 155025. `doi:10.1088/0264-9381/30/15/155025`. `arXiv:1304.0129 [hep-th]`.

[268] Moshe Rozali, Darren Smyth, Evgeny Sorkin and Jared B. Stang. 'Striped order in AdS/CFT correspondence'. *Phys. Rev.* D87.12 (2013), p. 126007. `doi:10.1103/PhysRevD.87.126007`. `arXiv:1304.3130 [hep-th]`.

[269] Benjamin Withers. 'The moduli space of striped black branes' (2013). `arXiv:1304.2011 [hep-th]`.

[270] Ning Bao, Sarah Harrison, Shamit Kachru and Subir Sachdev. 'Vortex Lattices and Crystalline Geometries'. *Phys. Rev.* D88.2 (2013), p. 026002. `doi:10.1103/PhysRevD.88.026002`. `arXiv:1303.4390 [hep-th]`.

[271] Niko Jokela, Matti Jarvinen and Matthew Lippert. 'Gravity dual of spin and charge density waves'. *JHEP* 12 (2014), p. 083. `doi:10.1007/JHEP12(2014)083`. `arXiv:1408.1397 [hep-th]`.

[272] Benjamin Withers. 'Holographic Checkerboards'. *JHEP* 09 (2014), p. 102. `doi:10.1007/JHEP09(2014)102`. `arXiv:1407.1085 [hep-th]`.

[273] Sera Cremonini and Annamaria Sinkovics. 'Spatially Modulated Instabilities of Geometries with Hyperscaling Violation'. *JHEP* 01 (2014), p. 099. `doi:10.1007/JHEP01(2014)099`. `arXiv:1212.4172 [hep-th]`.

[274] Aristomenis Donos, Jerome P. Gauntlett and Christiana Pantelidou. 'Magnetic and Electric AdS Solutions in String- and M-Theory'. *Class. Quant. Grav.* 29 (2012), p. 194006. `doi:10.1088/0264-9381/29/19/194006`. `arXiv:1112.4195 [hep-th]`.

[275] Yan-Yan Bu, Johanna Erdmenger, Jonathan P. Shock and Migael Strydom. 'Magnetic field induced lattice ground states from holography'. *JHEP* 03 (2013), p. 165. `doi:10.1007/JHEP03(2013)165`. `arXiv:1210.6669 [hep-th]`.

[276] Niko Jokela, Matti Jarvinen and Matthew Lippert. 'Holographic sliding stripes'. *Phys. Rev.* D95.8 (2017), p. 086006. `doi:10.1103/PhysRevD.95.086006`. `arXiv:1612.07323 [hep-th]`.

[277] Aristomenis Donos and Christiana Pantelidou. 'Holographic Magnetisation Density Waves'. *JHEP* 10 (2016), p. 038. `doi:10.1007/JHEP10(2016)038`. `arXiv:1607.01807 [hep-th]`.

[278] Aristomenis Donos and Jerome P. Gauntlett. 'Holographic Q-lattices'. *JHEP* 04 (2014), p. 040. doi:10.1007/JHEP04(2014)040. arXiv:1311.3292 [hep-th].

[279] Tomas Andrade and Benjamin Withers. 'A simple holographic model of momentum relaxation'. *JHEP* 05 (2014), p. 101. doi:10.1007/JHEP05(2014)101. arXiv:1311.5157 [hep-th].

[280] David Vegh. 'Holography without translational symmetry' (2013). arXiv:1301.0537 [hep-th].

[281] Richard A. Davison. 'Momentum relaxation in holographic massive gravity'. *Phys. Rev.* D88 (2013), p. 086003. doi:10.1103/PhysRevD.88.086003. arXiv:1306.5792 [hep-th].

[282] Mike Blake and David Tong. 'Universal Resistivity from Holographic Massive Gravity'. *Phys. Rev.* D88.10 (2013), p. 106004. doi:10.1103/PhysRevD.88.106004. arXiv:1308.4970 [hep-th].

[283] Mike Blake, David Tong and David Vegh. 'Holographic Lattices Give the Graviton an Effective Mass'. *Phys. Rev. Lett.* 112.7 (2014), p. 071602. doi:10.1103/PhysRevLett.112.071602. arXiv:1310.3832 [hep-th].

[284] Matteo Baggioli and Oriol Pujolas. 'Electron-Phonon Interactions, Metal-Insulator Transitions, and Holographic Massive Gravity'. *Phys. Rev. Lett.* 114.25 (2015), p. 251602. doi:10.1103/PhysRevLett.114.251602. arXiv:1411.1003 [hep-th].

[285] Aristomenis Donos and Jerome P. Gauntlett. 'The thermoelectric properties of inhomogeneous holographic lattices'. *JHEP* 1501 (2015), p. 035. doi:10.1007/JHEP01(2015)035. arXiv:1409.6875 [hep-th].

[286] Richard A. Davison and Blaise Goutéraux. 'Dissecting holographic conductivities'. *JHEP* 09 (2015), p. 090. doi:10.1007/JHEP09(2015)090. arXiv:1505.05092 [hep-th].

[287] Christopher P. Herzog, Pavel Kovtun, Subir Sachdev and Dam Thanh Son. 'Quantum critical transport, duality, and M-theory'. *Phys. Rev.* D75 (2007), p. 085020. doi:10.1103/PhysRevD.75.085020. arXiv:hep-th/0701036 [hep-th].

[288] Robert C. Myers, Subir Sachdev and Ajay Singh. 'Holographic Quantum Critical Transport without Self-Duality'. *Phys. Rev.* D83 (2011), p. 066017. doi:10.1103/PhysRevD.83.066017. arXiv:1010.0443 [hep-th].

[289] Aristomenis Donos and Jerome P. Gauntlett. 'Thermoelectric DC conductivities from black hole horizons'. *JHEP* 1411 (2014), p. 081. doi:10.1007/JHEP11(2014)081. arXiv:1406.4742 [hep-th].

[290] Aristomenis Donos and Jerome P. Gauntlett. 'Navier-Stokes Equations on Black Hole Horizons and DC Thermoelectric Conductivity'. *Phys. Rev.* D92.12 (2015), p. 121901. doi:10.1103/PhysRevD.92.121901. arXiv:1506.01360 [hep-th].

[291] Aristomenis Donos, Jerome P. Gauntlett, Tom Griffin and Luis Melgar. 'DC Conductivity and Higher Derivative Gravity'. *Class. Quant. Grav.* 34.13 (2017), p. 135015. doi:10.1088/1361-6382/aa744a. arXiv:1701.01389 [hep-th].

[292] Hai-Shan Liu, H. Lu and C. N. Pope. 'Holographic Heat Current as Noether Current'. *JHEP* 09 (2017), p. 146. doi:10.1007/JHEP09(2017)146. arXiv:1708.02329 [hep-th].

[293] Aristomenis Donos, Jerome P. Gauntlett, Tom Griffin and Luis Melgar. 'DC Conductivity of Magnetised Holographic Matter'. *JHEP* 01 (2016), p. 113. doi:10.1007/JHEP01(2016)113. arXiv:1511.00713 [hep-th].

[294] Elliot Banks, Aristomenis Donos, Jerome P. Gauntlett, Tom Griffin and Luis Melgar. 'Holographic thermal DC response in the hydrodynamic limit'. *Classical and Quantum Gravity* 34.4 (2017), p. 045001. doi:10.1088/1361-6382/aa51df. arXiv:1609.08912 [hep-th].

[295] D. A. Bandurin et al. 'Negative local resistance caused by viscous electron backflow in graphene'. *Science* 351 (Mar. 2016), pp. 1055–1058. doi:10.1126/science.aad0201. arXiv:1509.04165 [cond-mat.str-el].

[296] J. Crossno et al. 'Observation of the Dirac fluid and the breakdown of the Wiedemann-Franz law in graphene'. *Science* 351 (Mar. 2016), pp. 1058–1061. doi:10.1126/science.aad0343. arXiv:1509.04713 [cond-mat.mes-hall].

[297] P. J. W. Moll, P. Kushwaha, N. Nandi, B. Schmidt and A. P. Mackenzie. 'Evidence for hydrodynamic electron flow in $PdCoO_2$'. *Science* 351 (Mar. 2016), pp. 1061–1064. doi:10.1126/science.aac8385. arXiv:1509.05691 [cond-mat.str-el].

[298] H. Guo, E. Ilseven, G. Falkovich and L. S. Levitov. 'Higher-than-ballistic conduction of viscous electron flows'. *Proceedings of the National Academy of Science* 114 (Mar. 2017), pp. 3068–3073. doi:10.1073/pnas.1612181114. arXiv:1607.07269 [cond-mat.mes-hall].

[299] Andrew Lucas and Julia Steinberg. 'Charge diffusion and the butterfly effect in striped holographic matter'. *JHEP* 10 (2016), p. 143. doi:10.1007/JHEP10(2016)143. arXiv:1608.03286 [hep-th].

[300] Elliot Banks, Aristomenis Donos, Jerome P. Gauntlett, Tom Griffin and Luis Melgar. 'Thermal backflow in CFTs'. *Phys. Rev.* D95.2 (2017), p. 025022. doi:10.1103/PhysRevD.95.025022. arXiv:1610.00392 [hep-th].

[301] Luca V. Delacrétaz, Blaise Goutéraux, Sean A. Hartnoll and Anna Karlsson. 'Bad Metals from Fluctuating Density Waves'. *SciPost Phys.* 3 (2017), p. 025. doi:10.21468/SciPostPhys.3.3.025. arXiv:1612.04381 [cond-mat.str-el].

[302] Luca V. Delacrétaz, Blaise Goutéraux, Sean A. Hartnoll and Anna Karlsson. 'Theory of hydrodynamic transport in fluctuating electronic charge density wave states'. *Phys. Rev.* B96.19 (2017), p. 195128. doi:10.1103/PhysRevB.96.195128. arXiv:1702.05104 [cond-mat.str-el].

[303] Andrew Lucas and Sean A. Hartnoll. 'Resistivity bound for hydrodynamic bad metals'. *Proc. Nat. Acad. Sci.* 114 (2017), p. 1344. arXiv:1704.07384 [cond-mat.str-el].

[304] Vincenzo Scopelliti, Koenraad Schalm and Andrew Lucas. 'Hydrodynamic charge and heat transport on inhomogeneous curved spaces'. *Phys. Rev.* B96.7 (2017), p. 075150. doi:10.1103/PhysRevB.96.075150. arXiv:1705.04325 [cond-mat.str-el].

[305] A. H. Castro Neto, F. Guinea, N. M. R. Peres, K. S. Novoselov and A. K. Geim. 'The electronic properties of graphene'. *Rev. Mod. Phys.* 81 (1 2009), pp. 109–162. http://link.aps.org/doi/10.1103/RevModPhys.81.109doi: 10.1103/RevModPhys.81.109.

[306] F. M. D. Pellegrino, G. G. N. Angilella and R. Pucci. 'Strain effect on the optical conductivity of graphene'. *Phys. Rev. B* 81 (3 2010), p. 035411. http://link.aps.org/doi/10.1103/PhysRevB.81.035411doi: 10.1103/PhysRevB.81.035411.

[307] Chen Si, Zhimei Sun and Feng Liu. 'Strain engineering of graphene: a review'. *Nanoscale* 8 (6 2016), pp. 3207–3217. http://dx.doi.org/10.1039/C5NR07755Adoi:10.1039/C5NR07755A.

[308] J. Martin, N. Akerman, G. Ulbricht, T. Lohmann, J. H. Smet, K. von Klitzing and A. Yacoby. 'Observation of electron-hole puddles in graphene using a scanning single-electron transistor'. *Nature Physics* 4 (Feb. 2008), pp. 144–148. doi:10.1038/nphys781. arXiv:0705.2180 [cond-mat.mes-hall].

[309] E. Rossi and S. Das Sarma. 'Ground State of Graphene in the Presence of Random Charged Impurities'. *Physical Review Letters* 101.16, 166803 (Oct. 2008), p. 166803. doi:10.1103/PhysRevLett.101.166803. arXiv:0803.0963.

[310] Y. Zhang, V. W. Brar, C. Girit, A. Zettl and M. F. Crommie. 'Origin of spatial charge inhomogeneity in graphene'. *Nature Physics* 5 (Oct. 2009), pp. 722–726. doi:10.1038/nphys1365. arXiv:0902.4793 [cond-mat.mtrl-sci].

[311] J. Xue, J. Sanchez-Yamagishi, D. Bulmash, P. Jacquod, A. Deshpande, K. Watanabe, T. Taniguchi, P. Jarillo-Herrero and B. J. Leroy. 'Scanning tunnelling microscopy and spectroscopy of ultra-flat graphene on hexagonal boron nitride'. *Nature Materials* 10 (Apr. 2011), pp. 282–285. doi:10.1038/nmat2968. arXiv:1102.2642 [cond-mat.mtrl-sci].

[312] N. R. Cooper, B. I. Halperin and I. M. Ruzin. 'Thermoelectric response of an interacting two-dimensional electron gas in a quantizing magnetic field'. *Phys. Rev. B* 55 (4 Jan. 1997), pp. 2344–2359. http://link.aps.org/doi/10.1103/PhysRevB.55.2344doi:10.1103/PhysRevB.55.2344.

[313] Yi Ling, Peng Liu and Jian-Pin Wu. 'Holographic Butterfly Effect at Quantum Critical Points'. *JHEP* 10 (2017), p. 025. doi:10.1007/JHEP10(2017)025. arXiv:1610.02669 [hep-th].

[314] Richard A. Davison, Wenbo Fu, Antoine Georges, Yingfei Gu, Kristan Jensen and Subir Sachdev. 'Thermoelectric transport in disordered metals without quasiparticles: The Sachdev-Ye-Kitaev models and holography'. *Phys. Rev.* B95.15 (2017), p. 155131. doi:10.1103/PhysRevB.95.155131. arXiv:1612.00849 [cond-mat.str-el].

[315] Matteo Baggioli, Blaise Goutéraux, Elias Kiritsis and Wei-Jia Li. 'Higher derivative corrections to incoherent metallic transport in holography'. *JHEP* 03 (2017), p. 170. doi:10.1007/JHEP03(2017)170. arXiv:1612.05500 [hep-th].

[316] Shao-Feng Wu, Bin Wang, Xian-Hui Ge and Yu Tian. 'Collective diffusion and strange-metal transport' (2017). arXiv:1702.08803 [hep-th].

[317] Keun-Young Kim and Chao Niu. 'Diffusion and Butterfly Velocity at Finite Density'. *JHEP* 06 (2017), p. 030. doi:10.1007/JHEP06(2017)030. arXiv:1704.00947 [hep-th].

[318] Matteo Baggioli and Wei-Jia Li. 'Diffusivities bounds and chaos in holographic Horndeski theories'. *JHEP* 07 (2017), p. 055. doi:10.1007/JHEP07(2017)055. arXiv:1705.01766 [hep-th].

[319] Shao-Feng Wu, Bin Wang, Xian-Hui Ge and Yu Tian. 'Holographic RG flow of thermo-electric transports with momentum dissipation'. *Phys. Rev.* D97.6 (2018), p. 066029. doi:10.1103/PhysRevD.97.066029. arXiv:1706.00718 [hep-th].

[320] Aristomenis Donos, Jerome P. Gauntlett, Tom Griffin, Nakarin Lohitsiri and Luis Melgar. 'Holographic DC conductivity and Onsager relations'. *JHEP* 07 (2017), p. 006. doi:10.1007/JHEP07(2017)006. arXiv:1704.05141 [hep-th].

[321] Kedar Damle and Subir Sachdev. 'Nonzero-temperature transport near quantum critical points'. *Phys. Rev.* B56.14 (1997), p. 8714. doi:10.1103/PhysRevB.56.8714. arXiv:cond-mat/9705206 [cond-mat.str-el].

[322] Jan Zaanen. 'Superconductivity: Why the temperature is high'. *Nature* 430 (2004), p. 512.

[323] Dimitrios Giataganas, Umut Gürsoy and Juan F. Pedraza. 'Strongly-coupled anisotropic gauge theories and holography' (2017). arXiv:1708.05691 [hep-th].

[324] Andrew Lucas. 'Constraints on hydrodynamics from many-body quantum chaos' (2017). arXiv:1710.01005 [hep-th].

[325] Aristomenis Donos, Jerome P. Gauntlett, Christopher Rosen and Omar Sosa-Rodriguez. 'Boomerang RG flows with intermediate conformal invariance'. *JHEP* 04 (2018), p. 017. doi:10.1007/JHEP04(2018)017. arXiv:1712.08017 [hep-th].

[326] Igor L. Aleiner, Lara Faoro and Lev B. Ioffe. 'Microscopic model of quantum butterfly effect: out-of-time-order correlators and traveling combustion waves'. *Annals Phys.* 375 (2016), pp. 378–406. doi:10.1016/j.aop.2016.09.006. arXiv:1609.01251 [cond-mat.stat-mech].

[327] Yingfei Gu, Xiao-Liang Qi and Douglas Stanford. 'Local criticality, diffusion and chaos in generalized Sachdev-Ye-Kitaev models'. *JHEP* 05 (2017), p. 125. doi:10.1007/JHEP05(2017)125. arXiv:1609.07832 [hep-th].

[328] Aavishkar A. Patel and Subir Sachdev. 'Quantum chaos on a critical Fermi surface'. *Proc. Nat. Acad. Sci.* 114 (2017), pp. 1844–1849. doi:10.1073/pnas.1618185114. arXiv:1611.00003 [cond-mat.str-el].

[329] A. Bohrdt, C. B. Mendl, M. Endres and M. Knap. 'Scrambling and thermalization in a diffusive quantum many-body system'. *New J. Phys.* 19.6 (2017), p. 063001. doi:10.1088/1367-2630/aa719b. arXiv:1612.02434 [cond-mat.quant-gas].

[330] Yingfei Gu, Andrew Lucas and Xiao-Liang Qi. 'Energy diffusion and the butterfly effect in inhomogeneous Sachdev-Ye-Kitaev chains'. *SciPost Phys.* 2.3 (2017), p. 018. doi:10.21468/SciPostPhys.2.3.018. arXiv:1702.08462 [hep-th].

[331] Debanjan Chowdhury and Brian Swingle. 'Onset of many-body chaos in the $O(N)$ model'. *Phys. Rev.* D96.6 (2017), p. 065005. doi:10.1103/PhysRevD.96.065005. arXiv:1703.02545 [cond-mat.str-el].

[332] Aavishkar A. Patel, Debanjan Chowdhury, Subir Sachdev and Brian Swingle. 'Quantum butterfly effect in weakly interacting diffusive metals'. *Phys. Rev.* X7.3 (2017), p. 031047. doi:10.1103/PhysRevX.7.031047. arXiv:1703.07353 [cond-mat.str-el].

[333] Yochai Werman, Steven A. Kivelson and Erez Berg. 'Quantum chaos in an electron-phonon bad metal' (2017). arXiv:1705.07895 [cond-mat.str-el].

[334] Konstadinos Sfetsos. 'On gravitational shock waves in curved space-times'. *Nucl. Phys.* B436 (1995), pp. 721–745. doi:10.1016/0550-3213(94)00573-W. arXiv:hep-th/9408169 [hep-th].

[335] Romuald A. Janik, Jakub Jankowski and Hesam Soltanpanahi. 'Nonequilibrium Dynamics and Phase Transitions in Holographic Models'. *Phys. Rev. Lett.* 117.9 (2016), p. 091603. doi:10.1103/PhysRevLett.117.091603. arXiv:1512.06871 [hep-th].

[336] Romuald A. Janik, Jakub Jankowski and Hesam Soltanpanahi. 'Quasinormal modes and the phase structure of strongly coupled matter'. *JHEP* 06 (2016), p. 047. doi:10.1007/JHEP06(2016)047. arXiv:1603.05950 [hep-th].

[337] Anton F. Faedo, David Mateos, Christiana Pantelidou and Javier Tarrio. 'Towards a Holographic Quark Matter Crystal'. *JHEP* 10 (2017), p. 139. doi:10.1007/JHEP10(2017)139. arXiv:1707.06989 [hep-th].

[338] Romuald A. Janik, Jakub Jankowski and Hesam Soltanpanahi. 'Real-Time dynamics and phase separation in a holographic first order phase transition'. *Phys. Rev. Lett.* 119.26 (2017), p. 261601. doi:10.1103/PhysRevLett.119.261601. arXiv:1704.05387 [hep-th].

[339] Alex Buchel. 'A Holographic perspective on Gubser-Mitra conjecture'. *Nucl. Phys.* B731 (2005), pp. 109–124. doi:10.1016/j.nuclphysb.2005.10.014. arXiv:hep-th/0507275 [hep-th].

[340] Tomas Andrade, Matteo Baggioli, Alexander Krikun and Napat Poovuttikul. 'Pinning of longitudinal phonons in holographic spontaneous helices'. *JHEP* 02 (2018), p. 085. doi:10.1007/JHEP02(2018)085. arXiv:1708.08306 [hep-th].

[341] Lasma Alberte, Martin Ammon, Matteo Baggioli, Amadeo Jiménez and Oriol Pujolàs. 'Black hole elasticity and gapped transverse phonons in holography'. *JHEP* 01 (2018), p. 129. doi:10.1007/JHEP01(2018)129. arXiv:1708.08477 [hep-th].

[342] Niko Jokela, Matti Jarvinen and Matthew Lippert. 'Pinning of holographic sliding stripes'. *Phys. Rev.* D96.10 (2017), p. 106017. doi:10.1103/PhysRevD.96.106017. arXiv:1708.07837 [hep-th].

[343] Aristomenis Donos and Jerome P. Gauntlett. 'On the thermodynamics of periodic AdS black branes'. *JHEP* 1310 (2013), p. 038. doi:10.1007/JHEP10(2013)038. arXiv:1306.4937 [hep-th].

[344] Aristomenis Donos and Jerome P. Gauntlett. 'Minimally packed phases in holography'. *JHEP* 03 (2016), p. 148. doi:10.1007/JHEP03(2016)148. arXiv:1512.06861 [hep-th].

[345] Andrea Amoretti, Daniel Areán, Blaise Goutéraux and Daniele Musso. 'Effective holographic theory of quantum critical charge density waves' (2017). arXiv:1711.06610 [hep-th].

[346] Lasma Alberte, Martin Ammon, Matteo Baggioli, Amadeo Jiménez-Alba and Oriol Pujolas. 'Holographic Phonons' (2017). arXiv:1711.03100 [hep-th].

[347] Andrea Amoretti, Daniel Areán, Blaise Goutéraux and Daniele Musso. 'DC resistivity at holographic charge density wave quantum critical points' (2017). arXiv:1712.07994 [hep-th].

[348] Irene Amado, Matthias Kaminski and Karl Landsteiner. 'Hydrodynamics of Holographic Superconductors'. *JHEP* 05 (2009), p. 021. doi:10.1088/1126-6708/2009/05/021. arXiv:0903.2209 [hep-th].

[349] Christopher P. Herzog and Amos Yarom. 'Sound modes in holographic superfluids'. *Phys. Rev.* D80 (2009), p. 106002. doi:10.1103/PhysRevD.80.106002. arXiv:0906.4810 [hep-th].

[350] Irene Amado, Daniel Arean, Amadeo Jimenez-Alba, Karl Landsteiner, Luis Melgar and Ignacio Salazar Landea. 'Holographic Type II Goldstone bosons'. *JHEP* 07 (2013), p. 108. doi:10.1007/JHEP07(2013)108. arXiv:1302.5641 [hep-th].

[351] Ian Low and Aneesh V. Manohar. 'Spontaneously broken space-time symmetries and Goldstone's theorem'. *Phys. Rev. Lett.* 88 (2002), p. 101602. doi:10.1103/PhysRevLett.88.101602. arXiv:hep-th/0110285 [hep-th].

[352] Haruki Watanabe and Tomas Brauner. 'Spontaneous breaking of continuous translational invariance'. *Phys. Rev.* D85 (2012), p. 085010. doi:10.1103/PhysRevD.85.085010. arXiv:1112.3890 [cond-mat.stat-mech].

[353] Haruki Watanabe and Hitoshi Murayama. 'Redundancies in Nambu-Goldstone Bosons'. *Phys. Rev. Lett.* 110.18 (2013), p. 181601. doi:10.1103/PhysRevLett.110.181601. arXiv:1302.4800 [cond-mat.other].

[354] Lasma Alberte, Matteo Baggioli, Andrei Khmelnitsky and Oriol Pujolas. 'Solid Holography and Massive Gravity'. *JHEP* 02 (2016), p. 114. doi:10.1007/JHEP02(2016)114. arXiv:1510.09089 [hep-th].

[355] Frederik Denef and Sean A. Hartnoll. 'Landscape of superconducting membranes'. *Phys. Rev.* D79 (2009), p. 126008. doi:10.1103/PhysRevD.79.126008. arXiv:0901.1160 [hep-th].

[356] Aristomenis Donos, Jerome P. Gauntlett, Julian Sonner and Benjamin Withers. 'Competing orders in M-theory: superfluids, stripes and metamagnetism'. *JHEP* 03 (2013), p. 108. doi:10.1007/JHEP03(2013)108. arXiv:1212.0871 [hep-th].

[357] Eric D'Hoker and Per Kraus. 'Quantum Criticality via Magnetic Branes'. *Lect. Notes Phys.* 871 (2013), pp. 469–502. doi:10.1007/978-3-642-37305-3_18. arXiv:1208.1925 [hep-th].

[358] Elias Kiritsis and Li Li. 'Holographic Competition of Phases and Superconductivity'. *JHEP* 01 (2016), p. 147. doi:10.1007/JHEP01(2016)147. arXiv:1510.00020 [cond-mat.str-el].

[359] Elias Kiritsis, Francesco Nitti and Leandro Silva Pimenta. 'Exotic RG Flows from Holography'. *Fortsch. Phys.* 65.2 (2017), p. 1600120. doi:10.1002/prop.201600120. arXiv:1611.05493 [hep-th].

[360] U. Gürsoy, E. Kiritsis, F. Nitti and L. Silva Pimenta. 'Exotic holographic RG flows at finite temperature' (2018). arXiv:1805.01769 [hep-th].

[361] Aristomenis Donos, Jerome P. Gauntlett, Christopher Rosen and Omar Sosa-Rodriguez. 'Boomerang RG flows in M-theory with intermediate scaling'. *JHEP* 07 (2017), p. 128. doi:10.1007/JHEP07(2017)128. arXiv:1705.03000 [hep-th].

[362] Veronika E. Hubeny, Mukund Rangamani and Tadashi Takayanagi. 'A Covariant holographic entanglement entropy proposal'. *JHEP* 07 (2007), p. 062. `doi:10.1088/1126-6708/2007/07/062`. `arXiv:0705.0016 [hep-th]`.

[363] V. Balasubramanian, A. Bernamonti, J. de Boer, N. Copland, B. Craps, E. Keski-Vakkuri, B. Muller, A. Schafer, M. Shigemori and W. Staessens. 'Thermalization of Strongly Coupled Field Theories'. *Phys. Rev. Lett.* 106 (2011), p. 191601. `doi:10.1103/PhysRevLett.106.191601`. `arXiv:1012.4753 [hep-th]`.

[364] V. Balasubramanian, A. Bernamonti, J. de Boer, N. Copland, B. Craps, E. Keski-Vakkuri, B. Muller, A. Schafer, M. Shigemori and W. Staessens. 'Holographic Thermalization'. *Phys. Rev.* D84 (2011), p. 026010. `doi:10.1103/PhysRevD.84.026010`. `arXiv:1103.2683 [hep-th]`.

[365] Hong Liu and S. Josephine Suh. 'Entanglement Tsunami: Universal Scaling in Holographic Thermalization'. *Phys. Rev. Lett.* 112 (2014), p. 011601. `doi:10.1103/PhysRevLett.112.011601`. `arXiv:1305.7244 [hep-th]`.

[366] Hong Liu and S. Josephine Suh. 'Entanglement growth during thermalization in holographic systems'. *Phys. Rev.* D89.6 (2014), p. 066012. `doi:10.1103/PhysRevD.89.066012`. `arXiv:1311.1200 [hep-th]`.

[367] V. Balasubramanian, A. Bernamonti, J. de Boer, B. Craps, L. Franti, F. Galli, E. Keski-Vakkuri, B. Müller and A. Schäfer. 'Inhomogeneous holographic thermalization'. *JHEP* 10 (2013), p. 082. `doi:10.1007/JHEP10(2013)082`. `arXiv:1307.7086 [hep-th]`.

[368] Tim De Jonckheere and Jonathan Lindgren. 'Entanglement entropy in inhomogeneous quenches in AdS_3/CFT_2' (2018). `arXiv:1803.04718 [hep-th]`.

[369] Shinsei Ryu and Tadashi Takayanagi. 'Holographic derivation of entanglement entropy from AdS/CFT'. *Phys. Rev. Lett.* 96 (2006), p. 181602. `doi:10.1103/PhysRevLett.96.181602`. `arXiv:hep-th/0603001 [hep-th]`.

[370] Igor R. Klebanov, David Kutasov and Arvind Murugan. 'Entanglement as a probe of confinement'. *Nucl. Phys.* B796 (2008), pp. 274–293. `doi:10.1016/j.nuclphysb.2007.12.017`. `arXiv:0709.2140 [hep-th]`.

[371] Márk Mezei and Douglas Stanford. 'On entanglement spreading in chaotic systems'. *JHEP* 05 (2017), p. 065. `doi:10.1007/JHEP05(2017)065`. `arXiv:1608.05101 [hep-th]`.

[372] Márk Mezei. 'On entanglement spreading from holography'. *JHEP* 05 (2017), p. 064. `doi:10.1007/JHEP05(2017)064`. `arXiv:1612.00082 [hep-th]`.